J. Integer Sequences,
24 (2021), Article 21.3.7.

Nontrivial Effective Lower Bounds for the Least Common Multiple of a q-Arithmetic Progression

Bakir Farhi
Laboratoire de Mathématiques Appliquées
Faculté des Sciences Exactes
Université de Bejaia
06000 Bejaia
Algeria
bakir.farhi@gmail.com

Abstract

This paper is devoted to establish nontrivial effective lower bounds for the least common multiple of consecutive terms of a sequence $(u_n)_{n\in\mathbb{N}}$ whose general term has the form $u_n = r(q^n - 1)/(q - 1) + u_0$, where r, q and u_0 are non-negative integers satisfying some specific conditions. This can be considered as a q-analog of the lower bounds already obtained by the author (in 2005) and by Hong and Feng (in 2006) for arithmetic progressions.

1 Introduction and the main results

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers. For $t \in \mathbb{R}$, we let $\lfloor t \rfloor$ denote the integer part of t. We say that an integer a is a multiple of a non-zero rational number r if the quotient a/r is an integer. The letter q always denotes a positive integer; furthermore, it is assumed, if necessary, that $q \geq 2$. (This assumption is needed in Subsection 2.2.) Let us recall the standard notation of q-calculus (see, e.g., [10]). For

$n, k \in \mathbb{N}$, with $n \geq k$, we define

$$[n]_q := \frac{q^n - 1}{q - 1} \quad \text{for } q \neq 1 \text{ and } [n]_1 := n,$$

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q \quad (\text{with the convention } [0]_q! = 1),$$

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.$$

The numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$ are called the q-binomial coefficients (or the gaussian binomial coefficients) and it is well-known that they are all positive integers (see, e.g., [10]). From this last fact, we derive the important property stating that

$$\text{For all } a, b \in \mathbb{N}, \text{ the positive integer } [a]_q! [b]_q! \text{ divides the positive integer } [a+b]_q!. \tag{1}$$

Indeed, for $a, b \in \mathbb{N}$, we have $\frac{[a+b]_q!}{[a]_q! [b]_q!} = \left[\begin{smallmatrix} a+b \\ a \end{smallmatrix} \right]_q \in \mathbb{N}^*$.

The study of the least common multiple of consecutive positive integers began with Chebychev's work [4] in his attempts to prove the prime number theorem. The latter defined $\psi(n) := \log \mathrm{lcm}(1, 2, \dots, n)$ $(\forall n \geq 2)$ and showed that $\frac{\psi(n)}{n}$ is bounded between two positive constants, but he failed to prove that $\psi(n) \sim_{+\infty} n$, which is equivalent to the prime number theorem. Quite recently, Hanson [7] and Nair [12], respectively, obtained the bounds $\mathrm{lcm}(1, 2, \dots, n) \leq 3^n$ $(\forall n \in \mathbb{N}^*)$ and $\mathrm{lcm}(1, 2, \dots, n) \geq 2^n$ $(\forall n \geq 7)$ in simple and elegant ways. Later, the author [5, 6] obtained nontrivial effective lower bounds for the least common multiple of consecutive terms in an arithmetic progression. In particular, he proved that for any $u_0, r, n \in \mathbb{N}^*$, with $\gcd(u_0, r) = 1$, we have $\mathrm{lcm}(u_0, u_0 + r, \dots, u_0 + nr) \geq u_0 (r+1)^{n-1}$. By developing the author's method, Hong and Feng [8] managed to improve this lower bound to the optimal one:

$$\mathrm{lcm}(u_0, u_0 + r, \dots, u_0 + nr) \geq u_0 (r+1)^n \qquad (\forall n \in \mathbb{N}), \tag{2}$$

which was already conjectured by the author [5, 6]. It is interesting to note that the method used to obtain (2) is based on the following fundamental theorem:

Theorem 1 ([6, Theorem 2]). *Let I be a finite non-empty set of indices and $(u_i)_{i \in I}$ be a sequence of non-zero integers. Then the integer*

$$\mathrm{lcm}\left\{u_i,\ i \in I\right\} \cdot \mathrm{lcm}\left\{ \prod_{\substack{i \in I \\ i \neq j}} |u_i - u_j|,\ j \in I \right\}$$

is a multiple of the integer $\prod_{i \in I} u_i$.

Furthermore, several authors obtained improvements of (2) for n sufficiently large in terms of u_0 and r (see, e.g., [9, 11]). Concerning the asymptotic estimates and the effective upper bounds for the least common multiple of an arithmetic progression, we can cite the work of Bateman et al. [1] and the very recent work of Bousla [2].

In this paper, we apply and adapt the author's method [5, 6] (slightly developed by Hong and Feng [8]) to establish nontrivial effective lower bounds for the least common multiple of consecutive terms in a sequence that we call a *q-arithmetic progression*; that is, a sequence $(u_n)_n$ with general term having the form $u_n = r[n]_q + u_0$ ($\forall n \in \mathbb{N}$), where $r \in \mathbb{N}^*$, $u_0 \in \mathbb{N}$ and r, u_0, q satisfy some technical conditions. Our main results are the following:

Theorem 2 (The crucial result)**.** *Let q and r be two positive integers and u_0 be a non-negative integer. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of natural numbers whose general term u_n is given by $u_n = r[n]_q + u_0$. Suppose that $\gcd(u_0, r) = \gcd(u_1, q) = 1$. Then, for all positive integers n and k such that $n \geq k$, the positive integer $\operatorname{lcm}(u_k, u_{k+1}, \dots, u_n)$ is a multiple of the rational number $\frac{u_k u_{k+1} \cdots u_n}{[n-k]_q!}$.*

Theorem 3. *In the situation of Theorem 2, set*

$$A := \max\left(0\ ,\ \frac{u_0(q-1)+1-r}{2r}\right).$$

Then, for all positive integers n, we have

$$\operatorname{lcm}(u_1, u_2, \dots, u_n) \geq u_1 \left(\frac{r+1}{\sqrt{r}(A+1)}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}}.$$

Theorem 4. *In the situation of Theorem 2, set*

$$B := \max\left(r\ ,\ \frac{u_0(q-1)+1-r}{2}\right).$$

Then, for all positive integers n, we have

$$\operatorname{lcm}(u_1, u_2, \dots, u_n) \geq u_1 \left(\frac{r+1}{2\sqrt{B}}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}}.$$

Note that Theorem 2 is a q-analog of a result due to the author (see [5, Théorème 2.3] or [6, Theorem 3]). Furthermore, Theorems 3 and 4 are derived from Theorem 2 by optimizing a certain specific expression, and they can be considered as q-analogs of the results by the author [5, 6] and those by Hong and Feng [8].

From Theorems 3 and 4, we immediately derive the following two corollaries:

Corollary 5. *Let q, a and b be integers such that $q \geq 2$, $a \geq 1$ and $b \geq -a$ and let $(v_n)_{n\in\mathbb{N}}$ be the sequence of natural numbers whose general term v_n is given by*

$$v_n = aq^n + b \qquad (\forall n \in \mathbb{N}).$$

Suppose that $\gcd(aq, b) = \gcd(a + b, q - 1) = 1$ *and set*

$$A' := \max\left(0 \ , \ \frac{b}{2a} + \frac{1}{2a(q-1)}\right).$$

Then, for all positive integers n, we have

$$\operatorname{lcm}\left(v_1, v_2, \dots, v_n\right) \geq (aq+b)\left(\frac{a(q-1)+1}{\sqrt{a(q-1)}(A'+1)}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}}.$$

Corollary 6. *In the situation of Corollary 5, set*

$$B' := \max\left(a(q-1) \ , \ \frac{b(q-1)+1}{2}\right).$$

Then, for all positive integers n, we have

$$\operatorname{lcm}\left(v_1, v_2, \dots, v_n\right) \geq (aq+b)\left(\frac{a(q-1)+1}{2\sqrt{B'}}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}}.$$

2 The proofs

Throughout the following, we fix $q, r \in \mathbb{N}^*$ and $u_0 \in \mathbb{N}$ such that $\gcd(u_0, r) = \gcd(u_1, q) = 1$ and we let $(u_n)_{n \in \mathbb{N}}$ denote the sequence of natural numbers defined by its general term $u_n := r[n]_q + u_0$ ($\forall n \in \mathbb{N}$).

2.1 Proof of Theorem 2

To prove Theorem 2, we need the following three lemmas:

Lemma 7. *For all $i, j \in \mathbb{N}$, we have*

$$|u_i - u_j| = rq^{\min(i,j)}[|i-j|]_q.$$

Proof. Let $i, j \in \mathbb{N}$. Because the two sides of the equality of the lemma are both symmetric (in i and j), we may suppose without loss of generality that $i \geq j$. Doing so we have

$$\begin{aligned} |u_i - u_j| = u_i - u_j &= \left(r[i]_q + u_0\right) - \left(r[j]_q + u_0\right) \\ &= r\left([i]_q - [j]_q\right) \\ &= r\left(\frac{q^i - 1}{q-1} - \frac{q^j - 1}{q-1}\right) \end{aligned}$$

$$
\begin{aligned}
&= r\left(\frac{q^i - q^j}{q-1}\right) \\
&= rq^j\left(\frac{q^{i-j} - 1}{q-1}\right) \\
&= rq^j[i-j]_q \\
&= rq^{\min(i,j)}[|i-j|]_q,
\end{aligned}
$$

as required. The lemma is proved. $\square$

Lemma 8. *For all $n \in \mathbb{N}$, we have*

$$\gcd(u_n, r) = 1.$$

If in addition $n \geq 1$, then we have

$$\gcd(u_n, q) = 1.$$

Proof. Let $n \in \mathbb{N}$ and let us show that $\gcd(u_n, r) = 1$. This is equivalent to show that $d = 1$ is the only positive common divisor of u_n and r. So let d be a positive common divisor of u_n and r and let us show that $d = 1$. The hypotheses $d|u_n$ and $d|r$ together imply that $d|(u_n - r[n]_q) = u_0$. Hence d is a positive common divisor of u_0 and r. But since $\gcd(u_0, r) = 1$, it follows that $d = 1$, as required. Consequently, we have $\gcd(u_n, r) = 1$.

Next, let $n \in \mathbb{N}^*$ and let us show that $\gcd(u_n, q) = 1$. Equivalently, we have to show that $d = 1$ is the only positive common divisor of u_n and q. So let d be a positive common divisor of u_n and q and let us show that $d = 1$. The hypotheses $d|u_n$ and $d|q$ together imply that $d|((rq^n + u_0 q) - (q-1)u_n) = r + u_0 = u_1$. So d is a positive common divisor of u_1 and q. But since $\gcd(u_1, q) = 1$, we conclude that $d = 1$, as required. Consequently, we have $\gcd(u_n, q) = 1$. This completes the proof of the lemma. $\square$

Lemma 9. *For all positive integers n and k such that $n \geq k$ and any $j \in \{k, k+1, \dots, n\}$, we have*

$$\sum_{\substack{k \leq i \leq n \\ i \neq j}} \min(i,j) \leq \frac{(n-k)(n+k-1)}{2}.$$

Proof. Let n and k be positive integers such that $n \geq k$ and let $j \in \{k, k+1, \dots, n\}$. We have

$$
\begin{aligned}
\sum_{\substack{k \leq i \leq n \\ i \neq j}} \min(i,j) &= \sum_{k \leq i < j} \min(i,j) + \sum_{j < i \leq n} \min(i,j) \\
&= \sum_{k \leq i < j} i + \sum_{j < i \leq n} j
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(j-k)(j+k-1)}{2} + (n-j)j \\
&= \frac{2nj - j^2 - k^2 - j + k}{2} \\
&= \frac{(n-k)(n+k-1) + (n-j) - (n-j)^2}{2} \\
&\leq \frac{(n-k)(n+k-1)}{2}
\end{aligned}
$$

(since $n - j \leq (n-j)^2$, because $n - j \in \mathbb{N}$). The lemma is proved. $\square$

Now we are ready to prove the crucial Theorem 2:

Proof of Theorem 2. Let n and k be positive integers such that $n \geq k$. By applying the fundamental Theorem 1 to the set of indices $I = \{k, k+1, \dots, n\}$ and the sequence $(u_i)_{i \in I} = \{u_k, u_{k+1}, \dots, u_n\}$, we find that the positive integer

$$
\mathrm{lcm}\,(u_k, u_{k+1}, \dots, u_n) \cdot \mathrm{lcm} \left\{ \prod_{\substack{k \leq i \leq n \\ i \neq j}} |u_i - u_j| \, ; \; j = k, \dots, n \right\}
$$

is a multiple of the positive integer $u_k u_{k+1} \cdots u_n$. Now let us find a simple multiple for the positive integer $\mathrm{lcm} \left\{ \prod_{k \leq i \leq n, i \neq j} |u_i - u_j| ; j = k, \dots, n \right\}$. According to Lemma 7, we have for any $j \in \{k, k+1, \dots, n\}$, that

$$
\begin{aligned}
\prod_{\substack{k \leq i \leq n \\ i \neq j}} |u_i - u_j| &= \prod_{\substack{k \leq i \leq n \\ i \neq j}} \left(r q^{\min(i,j)} [|i-j|]_q \right) \\
&= r^{n-k} q^{\sum_{\substack{k \leq i \leq n \\ i \neq j}} \min(i,j)} \prod_{\substack{k \leq i \leq n \\ i \neq j}} [|i-j|]_q \\
&= r^{n-k} q^{\sum_{\substack{k \leq i \leq n \\ i \neq j}} \min(i,j)} [1]_q [2]_q \cdots [j-k]_q \times [1]_q [2]_q \cdots [n-j]_q \\
&= r^{n-k} q^{\sum_{\substack{k \leq i \leq n \\ i \neq j}} \min(i,j)} [j-k]_q! [n-j]_q!,
\end{aligned}
$$

which divides (according to Lemma 9 and Property (1)) the positive integer

$$
r^{n-k} q^{\frac{(n-k)(n+k-1)}{2}} [n-k]_q!.
$$

Consequently, the positive integer $\mathrm{lcm}\{\prod_{k \leq i \leq n, i \neq j} |u_i - u_j| ; j = k, \dots, n\}$ divides the positive integer $r^{n-k} q^{\frac{(n-k)(n+k-1)}{2}} [n-k]_q!$. It follows (according to what obtained at the beginning of this proof) that the positive integer $u_k u_{k+1} \cdots u_n$ divides the positive integer

$r^{n-k}q^{\frac{(n-k)(n+k-1)}{2}}[n-k]_q!\,\mathrm{lcm}\,(u_k, u_{k+1}, \dots, u_n)$. Next, since (according to Lemma 8) the integers u_i $(i \geq 1)$ are all coprime with r and q then the product $u_k u_{k+1} \cdots u_n$ is coprime with $r^{n-k}q^{\frac{(n-k)(n+k-1)}{2}}$, which shows (according to the Gauss lemma) that $u_k u_{k+1} \cdots u_n$ divides $[n-k]_q!\,\mathrm{lcm}\,(u_k, u_{k+1}, \dots, u_n)$. Equivalently, the positive integer $\mathrm{lcm}\,(u_k, u_{k+1}, \dots, u_n)$ is a multiple of the rational number $\frac{u_k u_{k+1} \cdots u_n}{[n-k]_q!}$. This completes the proof. $\square$

2.2 Proofs of Theorems 3 and 4 and their corollaries

To deduce Theorems 3 and 4 from Theorem 2, we need some additional preparations. Since, for $q = 1$, Theorems 3 and 4 are immediate consequences of (2), we may suppose for the sequel that $q \geq 2$. Next, we naturally extend the definition of u_n to negative indices n and for all $n, k \in \mathbb{Z}$ such that $n \geq k$ we define

$$C_{n,k} := \frac{u_k u_{k+1} \cdots u_n}{[n-k]_q!}.$$

Furthermore, for a given positive integer n, the problem of determining the positive integer $k \leq n$ which maximizes $C_{n,k}$ leads us to introduce the function $f : \mathbb{R} \rightarrow \mathbb{R}$, defined as follows:

$$f(x) := q^{x-1}\left(rq^{x-1} + u_0(q-1) + 1 - r\right) \qquad (\forall x \in \mathbb{R}).$$

It is immediate that f increases, tends to 0 as x tends to $(-\infty)$ and satisfies, for all $n \in \mathbb{N}^*$, the property

$$\forall k \in \mathbb{Z} : \quad k > n \Rightarrow f(k) > q^n.$$

For a given positive integer n, these properties ensure the existence of a largest $k_n \in \mathbb{Z}$ satisfying $f(k_n) \leq q^n$, and show, in addition, that $k_n \leq n$. From the increase of f and the definition of k_n $(n \in \mathbb{N}^*)$, we derive that

$$\forall k \in \mathbb{Z} : \quad k \leq k_n \Longleftrightarrow f(k) \leq q^n. \tag{3}$$

Now since for any $n \in \mathbb{N}^*$ and any $k \in \mathbb{Z}$, we have

$$\begin{aligned} f(k) \leq q^n &\Longleftrightarrow q^{k-1}\left(rq^{k-1} + u_0(q-1) + 1 - r\right) \leq q^n \\ &\Longleftrightarrow rq^{k-1} + u_0(q-1) + 1 - r \leq q^{n-k+1} \\ &\Longleftrightarrow \frac{q^{n-k+1} - 1}{q-1} \geq r\frac{q^{k-1} - 1}{q-1} + u_0 \\ &\Longleftrightarrow [n-k+1]_q \geq u_{k-1}, \end{aligned}$$

then Property (3) is equivalent to

$$\forall k \in \mathbb{Z} : \quad k \leq k_n \Longleftrightarrow [n-k+1]_q \geq u_{k-1}. \tag{4}$$

For a given positive integer n, we set

$$\ell_n := \max(1, k_n).$$

Since $k_n \leq n$, we have that $\ell_n \in \{1, 2, \ldots, n\}$.

Next, it is immediate that f satisfies the following inequality:

$$f(x-1) \leq \frac{1}{q} f(x) \qquad (\forall x \in \mathbb{R}). \tag{5}$$

For a fixed $n \in \mathbb{N}^*$, the following lemmas aim to maximize the quantity $C_{n,k}$ $(1 \leq k \leq n)$ appearing in Theorem 2. Precisely, we shall determine two simple upper bounds for $\max_{1 \leq k \leq n} C_{n,k}$ from which we derive our Theorems 3 and 4.

Lemma 10. *Let n be a fixed positive integer. The sequence $(C_{n,k})_{k \in \mathbb{Z}, k \leq n}$ is non-decreasing until $k = k_n$, and then it decreases. So it reaches its maximal value at $k = k_n$.*

Proof. For any $k \in \mathbb{Z}$, with $k \leq n$, we have

$$\begin{aligned}
C_{n,k} \geq C_{n,k-1} &\Longleftrightarrow \frac{C_{n,k}}{C_{n,k-1}} \geq 1 \\
&\Longleftrightarrow \frac{u_k u_{k+1} \cdots u_n}{[n-k]_q!} \Big/ \frac{u_{k-1} u_k \cdots u_n}{[n-k+1]_q!} \geq 1 \\
&\Longleftrightarrow \frac{[n-k+1]_q}{u_{k-1}} \geq 1 \\
&\Longleftrightarrow [n-k+1]_q \geq u_{k-1} \\
&\Longleftrightarrow k \leq k_n \qquad \text{(according to (4))},
\end{aligned}$$

which concludes the proof. $\square$

From the last lemma, we obviously derive the following:

Lemma 11. *Let n be a fixed positive integer. Then the sequence $(C_{n,k})_{1 \leq k \leq n}$ reaches its maximal value at $k = \ell_n$.* $\square$

If $n \in \mathbb{N}^*$ is fixed, we have from Lemma 11 above that $\max_{1 \leq k \leq n} C_{n,k} = C_{n,\ell_n}$; however, the exact value of C_{n,ℓ_n} (in terms of n, q, r, u_0) is complicated. The lemmas below provide studies of the sequences $(k_n)_n$, $(\ell_n)_n$ and $(C_{n,\ell_n})_n$ in order to find a good lower bound for C_{n,ℓ_n} that has a simple expression in terms of n, q, r, u_0.

Lemma 12. *For all positive integers n, we have*

$$k_n \leq k_{n+1} \leq k_n + 1.$$

In other words, we have

$$k_{n+1} \in \{k_n, k_n + 1\}.$$

Proof. Let n be a fixed positive integer. By definition of the integer k_n, we have

$$f(k_n) \leq q^n \leq q^{n+1},$$

which implies (by definition of the integer k_{n+1}) that

$$k_{n+1} \geq k_n.$$

On the other hand, we have (according to (5) and the definition of the integer k_{n+1})

$$f(k_{n+1} - 1) \leq \frac{1}{q} f(k_{n+1}) \leq \frac{1}{q}\, q^{n+1} = q^n,$$

which implies (by definition of the integer k_n) that

$$k_n \geq k_{n+1} - 1;$$

that is

$$k_{n+1} \leq k_n + 1.$$

This completes the proof of the lemma. $\square$

Lemma 13. *For all positive integers n, we have*

$$\ell_{n+1} \in \{\ell_n, \ell_n + 1\}.$$

In addition, in the case when $\ell_{n+1} = \ell_n + 1$, we have $\ell_n = k_n$ and $\ell_{n+1} = k_{n+1} = k_n + 1$.

Proof. Let n be a fixed positive integer. By Lemma 12, we have that

$$k_n \leq k_{n+1} \leq k_n + 1.$$

Hence

$$\max(1, k_n) \leq \max(1, k_{n+1}) \leq \max(1, k_n + 1) = \max(0, k_n) + 1 \leq \max(1, k_n) + 1;$$

therefore

$$\ell_n \leq \ell_{n+1} \leq \ell_n + 1.$$

This confirms the first part of the lemma.

Now let us show the second part of the lemma. So suppose that $\ell_{n+1} = \ell_n + 1$ and show that $\ell_n = k_n$ and $\ell_{n+1} = k_{n+1} = k_n + 1$. Since $\ell_n = \max(1, k_n) \geq 1$ and $\ell_{n+1} = \ell_n + 1$ then $\ell_{n+1} \geq 2$. This implies that $\ell_{n+1} \neq 1$; thus $\ell_{n+1} = k_{n+1}$ (since $\ell_{n+1} = \max(1, k_{n+1}) \in \{1, k_{n+1}\}$). Using this and Lemma 12 above, we derive that $\ell_n = \ell_{n+1} - 1 = k_{n+1} - 1 \leq (k_n + 1) - 1 = k_n$; that is $\ell_n \leq k_n$. But since $\ell_n = \max(1, k_n) \geq k_n$, we conclude that $\ell_n = k_n$. This completes the proof of the second part of the lemma and finishes the proof. $\square$

Lemma 14. *For all positive integers n, we have*

$$C_{n+1,\ell_{n+1}} \geq (r+1)q^{\ell_n - 1} C_{n,\ell_n}.$$

Proof. Let n be a fixed positive integer. By Lemma 13, we have that $\ell_{n+1} \in \{\ell_n, \ell_{n+1}\}$. So we have to distinguish two cases:

Case 1: (if $\ell_{n+1} = \ell_n$)
In this case, we have

$$\begin{aligned} C_{n+1,\ell_{n+1}} = C_{n+1,\ell_n} = \frac{u_{\ell_n} u_{\ell_n+1} \cdots u_n u_{n+1}}{[n+1-\ell_n]_q!} &= \frac{u_{\ell_n} u_{\ell_n+1} \cdots u_n}{[n-\ell_n]_q!} \cdot \frac{u_{n+1}}{[n+1-\ell_n]_q} \\ &= C_{n,\ell_n} \cdot \frac{u_{n+1}}{[n+1-\ell_n]_q}. \end{aligned} \tag{6}$$

Next, we have

$$\begin{aligned} u_{n+1} - (r+1)q^{\ell_n-1}[n+1-\ell_n]_q &= r[n+1]_q + u_0 - (r+1)q^{\ell_n-1}\left(\frac{q^{n+1-\ell_n}-1}{q-1}\right) \\ &= r\left(\frac{q^{n+1}-1}{q-1}\right) + u_0 - (r+1)\left(\frac{q^n - q^{\ell_n-1}}{q-1}\right) \\ &= \frac{r(q^{n+1}-1) + u_0(q-1) - (r+1)(q^n - q^{\ell_n-1})}{q-1} \\ &= \frac{rq^{n+1} - (r+1)q^n + (r+1)q^{\ell_n-1} - r + u_0(q-1)}{q-1} \\ &= \frac{\big(r(q-1)-1\big)q^n + [(r+1)q^{\ell_n-1} - r] + u_0(q-1)}{q-1} \\ &\geq 0 \end{aligned}$$

(since $q \geq 2$, $r \geq 1$, $u_0 \geq 0$ and $\ell_n \geq 1$). Thus

$$\frac{u_{n+1}}{[n+1-\ell_n]_q} \geq (r+1)q^{\ell_n-1}.$$

By substituting this into (6), we get

$$C_{n+1,\ell_{n+1}} \geq (r+1)q^{\ell_n-1} C_{n,\ell_n},$$

as required.

Case 2: (if $\ell_{n+1} = \ell_n + 1$)

In this case, we have (according to Lemma 13): $\ell_n = k_n$ and $\ell_{n+1} = k_{n+1} = k_n + 1$. Thus, we have

$$C_{n+1,\ell_{n+1}} = C_{n+1,k_n+1} = \frac{u_{k_n+1}u_{k_n+2}\cdots u_n u_{n+1}}{[n-k_n]_q!} = C_{n,k_n} \cdot \frac{u_{n+1}}{u_{k_n}} = C_{n,\ell_n} \cdot \frac{u_{n+1}}{u_{k_n}}. \tag{7}$$

Next, according to the inequality of the right-hand side of (4) (applied for $(n+1)$ instead of n and k_{n+1} instead of k), we have (since $k_{n+1} \leq k_{n+1}$)

$$u_{k_n} = u_{k_{n+1}-1} \leq [(n+1) - k_{n+1} + 1]_q = [n - k_n + 1]_q.$$

Hence

$$\begin{aligned}
u_{n+1} - (r+1)q^{\ell_n - 1}u_{k_n} &= u_{n+1} - (r+1)q^{k_n-1}u_{k_n} \\
&\geq u_{n+1} - (r+1)q^{k_n-1}[n-k_n+1]_q \\
&= r\left(\frac{q^{n+1}-1}{q-1}\right) + u_0 - (r+1)q^{k_n-1}\left(\frac{q^{n-k_n+1}-1}{q-1}\right) \\
&= \frac{r(q^{n+1}-1) + u_0(q-1) - (r+1)(q^n - q^{k_n-1})}{q-1} \\
&= \frac{(r(q-1)-1)q^n + u_0(q-1) + (r+1)q^{k_n-1} - r}{q-1} \\
&\geq 0
\end{aligned}$$

(since $q \geq 2$, $r \geq 1$, $u_0 \geq 0$ and $k_n = \ell_n \geq 1$). Thus

$$\frac{u_{n+1}}{u_{k_n}} \geq (r+1)q^{\ell_n-1}.$$

By substituting this into (7), we get

$$C_{n+1,\ell_{n+1}} \geq (r+1)q^{\ell_n-1}C_{n,\ell_n},$$

as required. The proof of the lemma is complete. □

By induction, we derive the following from Lemma 14 above:

Corollary 15. *For all positive integers n, we have*

$$C_{n,\ell_n} \geq u_1(r+1)^{n-1}q^{\sum_{i=1}^{n-1}(\ell_i-1)}.$$

Proof. Let n be a positive integer. From Lemma 14, we have

$$C_{n,\ell_n} = C_{1,\ell_1}\prod_{i=1}^{n-1}\frac{C_{i+1,\ell_{i+1}}}{C_{i,\ell_i}} \geq C_{1,\ell_1}\prod_{i=1}^{n-1}\left\{(r+1)q^{\ell_i-1}\right\} = C_{1,\ell_1}(r+1)^{n-1}q^{\sum_{i=1}^{n-1}(\ell_i-1)}.$$

Next, since $k_1 \leq 1$, we have $\ell_1 = \max(1, k_1) = 1$; hence $C_{1,\ell_1} = C_{1,1} = \frac{u_1}{[0]_q!} = u_1$. Consequently, we have

$$C_{n,\ell_n} \geq u_1(r+1)^{n-1}q^{\sum_{i=1}^{n-1}(\ell_i-1)},$$

as required. The corollary is proved. □

From Theorem 2 and Corollary 15 above, we immediately deduce the following result:

Corollary 16. *For all positive integers n, we have*

$$\operatorname{lcm}(u_1, u_2, \ldots, u_n) \geq u_1(r+1)^{n-1} q^{\sum_{i=1}^{n-1}(\ell_i - 1)}.$$

Proof. Let n be a fixed positive integer. Since the positive integer $\operatorname{lcm}(u_1, u_2, \ldots, u_n)$ is obviously a multiple of the positive integer $\operatorname{lcm}(u_{\ell_n}, u_{\ell_n+1}, \ldots, u_n)$, which is a multiple of the rational number $\frac{u_{\ell_n} u_{\ell_n+1} \cdots u_n}{[n-\ell_n]_q!} = C_{n,\ell_n}$ (according to Theorem 2), then we have

$$\operatorname{lcm}(u_1, u_2, \ldots, u_n) \geq C_{n,\ell_n}.$$

The result of the corollary then follows from Corollary 15. The proof is complete. □

Remark 17. If we allow to take $q = 1$ in Corollary 16, then we exactly obtain the result of Hong and Feng [8] (recalled in (2)).

Now in order to derive an explicit lower bound for $\operatorname{lcm}(u_1, u_2, \ldots, u_n)$ $(n \geq 1)$ from Corollary 16 above, it remains to bound the ℓ_i from below in terms of n, q, r and u_0. Here we just give two ways to bound the ℓ_i from below, but there are certainly other ways (perhaps more intelligent) to do this. We have the following lemmas:

Lemma 18. *Let*

$$A := \max\left(0\ ,\ \frac{u_0(q-1)+1-r}{2r}\right).$$

Then, for all positive integers n, we have

$$\ell_n > \frac{1}{2}\left(n - \frac{\log r + 2\log(A+1)}{\log q}\right).$$

Proof. Let n be a fixed positive integer. Since the inequality of the lemma is obvious for $n \leq \frac{\log r + 2\log(A+1)}{\log q}$, we may assume for the sequel that $n > \frac{\log r + 2\log(A+1)}{\log q}$. Now for any $x \geq 1$, we have

$$\begin{aligned}
f(x) &:= q^{x-1}\left(rq^{x-1} + u_0(q-1) + 1 - r\right) \\
&= r\left(\left(q^{x-1} + \frac{u_0(q-1)+1-r}{2r}\right)^2 - \left(\frac{u_0(q-1)+1-r}{2r}\right)^2\right) \\
&\leq r\left(q^{x-1} + \frac{u_0(q-1)+1-r}{2r}\right)^2 \\
&\leq r\left(q^{x-1} + A\right)^2 \\
&\leq r\left(q^{x-1} + Aq^{x-1}\right)^2 \\
&= r(A+1)^2 q^{2(x-1)}.
\end{aligned}$$

By applying this for

$$x_0 := \frac{1}{2}\left(n - \frac{\log r + 2\log(A+1)}{\log q}\right) + 1$$

(which is > 1 according to our assumption $n > \frac{\log r + 2\log(A+1)}{\log q}$), we get

$$f(x_0) \leq r(A+1)^2 q^{n - \frac{\log r + 2\log(A+1)}{\log q}} = q^n.$$

Then, since f is increasing and $\lfloor x_0 \rfloor \leq x_0$, we derive that

$$f(\lfloor x_0 \rfloor) \leq f(x_0) \leq q^n,$$

which implies (according to the definition of k_n) that

$$k_n \geq \lfloor x_0 \rfloor > x_0 - 1.$$

Hence

$$\ell_n := \max(1, k_n) \geq k_n > x_0 - 1,$$

that is

$$\ell_n > \frac{1}{2}\left(n - \frac{\log r + 2\log(A+1)}{\log q}\right),$$

as required. The lemma is proved. $\square$

Lemma 19. *Let*

$$B := \max\left(r\ ,\ \frac{u_0(q-1) + 1 - r}{2}\right).$$

Then, for all positive integers n, we have

$$\ell_n > \frac{1}{2}\left(n - \frac{\log(4B)}{\log q}\right).$$

Proof. Let n be a fixed positive integer. Since the inequality of the lemma is obvious for $n \leq \frac{\log(4B)}{\log q}$, we may assume for the sequel that $n > \frac{\log(4B)}{\log q}$. Now for any $x \geq 1$, we have

$$\begin{aligned} f(x) &:= q^{x-1}\left(rq^{x-1} + u_0(q-1) + 1 - r\right) \\ &\leq q^{x-1}\left(Bq^{x-1} + 2B\right) \\ &< B\left(q^{x-1} + 1\right)^2 \\ &\leq B\left(2q^{x-1}\right)^2 \\ &= 4Bq^{2(x-1)}. \end{aligned}$$

By applying this for

$$x_1 := \frac{1}{2}\left(n - \frac{\log(4B)}{\log q}\right) + 1$$

(which is > 1 according to our assumption $n > \frac{\log(4B)}{\log q}$), we get

$$f(x_1) \leq 4Bq^{n - \frac{\log(4B)}{\log q}} = q^n.$$

Then, since f is increasing and $\lfloor x_1 \rfloor \leq x_1$, we derive that

$$f(\lfloor x_1 \rfloor) \leq f(x_1) \leq q^n,$$

which implies (according to the definition of k_n) that

$$k_n \geq \lfloor x_1 \rfloor > x_1 - 1 = \frac{1}{2}\left(n - \frac{\log(4B)}{\log q}\right).$$

Hence

$$\ell_n := \max(1, k_n) \geq k_n > \frac{1}{2}\left(n - \frac{\log(4B)}{\log q}\right),$$

as required. The lemma is proved. $\square$

We are now ready to prove Theorems 3 and 4 announced in Section 1.

Proof of Theorem 3. By using successively Corollary 16 and Lemma 18, we have for all $n \in \mathbb{N}^*$ that

$$\begin{aligned}
\operatorname{lcm}(u_1, u_2, \dots, u_n) &\geq u_1 (r+1)^{n-1} q^{\sum_{i=1}^{n-1} (\ell_i - 1)} \\
&\geq u_1 (r+1)^{n-1} q^{\frac{(n-1)(n-4)}{4} - \frac{1}{2} \frac{\log r + 2\log(A+1)}{\log q}(n-1)} \\
&= u_1 \left(\frac{r+1}{\sqrt{r}(A+1)}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}},
\end{aligned}$$

as required. $\square$

Proof of Theorem 4. By using successively Corollary 16 and Lemma 19, we have for all $n \in \mathbb{N}^*$

$$\begin{aligned}
\operatorname{lcm}(u_1, u_2, \dots, u_n) &\geq u_1 (r+1)^{n-1} q^{\sum_{i=1}^{n-1} (\ell_i - 1)} \\
&\geq u_1 (r+1)^{n-1} q^{\frac{(n-1)(n-4)}{4} - \frac{1}{2} \frac{\log(4B)}{\log q}(n-1)} \\
&= u_1 \left(\frac{r+1}{2\sqrt{B}}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}},
\end{aligned}$$

as required. $\square$

Proof of Corollary 5. It suffices to remark that $v_n = a(q-1)[n]_q + a + b$ ($\forall n \in \mathbb{N}$) and then to apply Theorem 3 for the sequence $(v_n)_{n\in\mathbb{N}}$. We just specify that the imposed conditions $\gcd(aq, b) = \gcd(a+b, q-1) = 1$ guarantee the conditions $\gcd(v_0, r) = \gcd(v_1, q) = 1$ required in Theorem 3 (with $r := a(q-1)$). $\square$

Proof of Corollary 6. We simply apply Theorem 4 for the sequence $(v_n)_{n\in\mathbb{N}}$, after noticing that its general term can be written as: $v_n = a(q-1)[n]_q + a + b$. $\square$

3 Numerical examples and remarks

By applying our main results, we get for example the following nontrivial effective estimates:

- $\mathrm{lcm}\,(2^1-1, 2^2-1, \dots, 2^n-1) \geq 2^{\frac{n(n-1)}{4}}$ $\quad(\forall n \geq 1)$
  (Apply Theorem 3 for $u_n = [n]_2 = 2^n - 1$).

- $\mathrm{lcm}\,(2^1+1, 2^2+1, \dots, 2^n+1) \geq 3 \cdot 2^{\frac{(n-1)(n-4)}{4}}$ $\quad(\forall n \geq 1)$
  (Apply one of the two corollaries 5 or 6 for $v_n = 2^n + 1$).

- $\mathrm{lcm}\,(3^1+1, 3^2+1, \dots, 3^n+1) \geq 4 \cdot 3^{\frac{(n-1)(n-4)}{4}}$ $\quad(\forall n \geq 1)$
  (Observe that $\mathrm{lcm}\,(3^1+1, 3^2+1, \dots, 3^n+1) = 2\,\mathrm{lcm}\left(\frac{3^1+1}{2}, \frac{3^2+1}{2}, \dots, \frac{3^n+1}{2}\right)$ and apply one of the two Theorems 3 or 4 for $u_n = [n]_3 + 1 = \frac{3^n+1}{2}$).

Remark 20.

(a) Theorems 3 and 4 are incomparable in the sense that there are situations where Theorem 3 is stronger than Theorem 4, and other situations where we have the converse. For example, it is easy to verify that if $u_0(q-1)+1-r \leq 0$, then Theorem 3 is stronger than Theorem 4, while if $u_0(q-1)+1-3r > 0$, then Theorem 4 is stronger than Theorem 3.

(b) By refining the arguments of bounding from below the ℓ_i (that is the arguments of the proofs of Lemmas 18 and 19), it is perhaps possible to obtain a lower bound for $\mathrm{lcm}\,(u_1, u_2, \dots, u_n)$ $(n \geq 1)$ of the form

$$\mathrm{lcm}\,(u_1, u_2, \dots, u_n) \geq c \left(\frac{r+1}{\sqrt{r}}\right)^{n-1} q^{\frac{(n-1)(n-4)}{4}},$$

where c is a positive constant depending only on q, r and u_0. It appears that this is the best that can be expected from this method!

(c) It is remarkable that our lower bounds of $\mathrm{lcm}\,(u_1, u_2, \dots, u_n)$, for the considered sequences $(u_n)_n$, are quite close to $\sqrt{u_1 u_2 \cdots u_n}$. More precisely, we can easily deduce from our main results that in the same context, we have $\mathrm{lcm}\,(u_1, u_2, \dots, u_n) \geq c_3 c_4^n \sqrt{u_1 u_2 \cdots u_n}$, for some suitable positive constants c_3 and c_4, depending only on q, r and u_0.

(d) There is something in common between our results and the recent result by Bousla and Farhi [3] providing effective bounds for $\mathrm{lcm}(U_1, U_2, \dots, U_n)$, when $(U_n)_{n\in\mathbb{N}}$ is a particular Lucas sequence; precisely, when $(U_n)_n$ is recursively defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = PU_{n+1} - QU_n$ $(\forall n \in \mathbb{N})$ for some $P, Q \in \mathbb{Z}^*$, with $P^2 - 4Q > 0$ and

$\gcd(P, Q) = 1$. Indeed, if we take $P = q + 1$ and $Q = q$ (for some integer $q \geq 2$), we obtain that $U_n = [n]_q$ and the Bousla-Farhi lower bound then gives

$$\operatorname{lcm}\left([1]_q, [2]_q, \dots, [n]_q\right) \;\geq\; q^{\frac{n^2}{4} - \frac{n}{2} - 1} \qquad (\forall n \geq 1),$$

which is almost the same as what we obtained in this paper.

References

[1] P. Bateman, J. Kalb, and A. Stenger, A limit involving least common multiples: 10797, *Amer. Math. Monthly* **109** (2002), 393–394.

[2] S. A. Bousla, Nontrivial upper bounds for the least common multiple of an arithmetic progression, preprint, 2020. Available at https://arxiv.org/abs/2004.07335.

[3] S. A. Bousla and B. Farhi, Identités et estimations concernant le plus petit commun multiple de suites à forte divisibilité, *C. R. Math. Acad. Sci. Paris* **358** (2020), 481–487.

[4] P. L. Chebyshev, Mémoire sur les nombres premiers, *J. Math. Pures Appl.* **17** (1852), 366–390.

[5] B. Farhi, Minorations non triviales du plus petit commun multiple de certaines suites finies d'entiers, *C. R. Acad. Sci. Paris, Sér. I* **341** (2005), 469–474.

[6] B. Farhi, Nontrivial lower bounds for the least common multiple of some finite sequences of integers, *J. Number Theory* **125** (2007), 393–411.

[7] D. Hanson, On the product of the primes, *Canad. Math. Bull.* **15** (1972), 33–37.

[8] S. Hong and W. Feng, Lower bounds for the least common multiple of finite arithmetic progressions, *C. R. Acad. Sci. Paris, Sér. I* **343** (2006), 695–698.

[9] S. Hong and S. D. Kominers, Further improvements of lower bounds for the least common multiples of arithmetic progressions, *Proc. Amer. Math. Soc.* **138** (2010), 809–813.

[10] V. Kac and P. Cheung, *Quantum Calculus,* Universitext, Springer-Verlag, 2002.

[11] D. Kane and S. D. Kominers, Asymptotic improvements of lower bounds for the least common multiples of arithmetic progressions, *Canad. Math. Bull.* **57** (2014), 551–561.

[12] M. Nair, On Chebyshev-type inequalities for primes, *Amer. Math. Monthly* **89** (1982), 126–129.

2010 *Mathematics Subject Classification*: Primary 11A05; Secondary 11B25, 11B65, 05A30.
Keywords: least common multiple, q-analog, arithmetic progression.

Received September 21 2020; revised version received February 17 2021. Published in *Journal of Integer Sequences*, February 17 2021.