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New results on the least common multiple of consecutive integers

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Abstract

When studying the least common multiple of some finite sequences of integers, the first author introduced the interesting arithmetic functions g_k $(k \in \mathbb{N})$, defined by $g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}(n,n+1,\dots,n+k)}$ $(\forall n \in \mathbb{N} \setminus \{0\})$. He proved that g_k $(k \in \mathbb{N})$ is periodic and $k!$ is a period of g_k. He raised the open problem consisting to determine the smallest positive period P_k of g_k. Very recently, S. Hong and Y. Yang have improved the period $k!$ of g_k to $\mathrm{lcm}(1,2,\dots,k)$. In addition, they have conjectured that P_k is always a multiple of the positive integer $\frac{\mathrm{lcm}(1,2,\dots,k,k+1)}{k+1}$. An immediate consequence of this conjecture states that if $(k+1)$ is prime then the exact period of g_k is precisely equal to $\mathrm{lcm}(1,2,\dots,k)$.

In this paper, we first prove the conjecture of S. Hong and Y. Yang and then we give the exact value of P_k $(k \in \mathbb{N})$. We deduce, as a corollary, that P_k is equal to the part of $\mathrm{lcm}(1,2,\dots,k)$ not divisible by some prime.

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1 Introduction

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers.

Many results concerning the least common multiple of sequences of integers are known. The most famous is nothing else than an equivalent of the prime number theorem; it sates that $\log \mathrm{lcm}(1,2,\dots,n) \sim n$ as n tends to infinity (see e.g., [6]). Effective bounds for $\mathrm{lcm}(1,2,\dots,n)$ are also given by several authors (see e.g., [5] and [10]).

Recently, the topic has undergone important developments. In [1], Bateman, Kalb and Stenger have obtained an equivalent for $\log \mathrm{lcm}(u_1, u_2, \dots, u_n)$ when $(u_n)_n$ is an arithmetic progression. In [2], Cilleruelo has obtained a simple equivalent for the least common multiple of a quadratic progression. For the effective bounds, Farhi [3] [4] got lower bounds for $\mathrm{lcm}(u_0, u_1, \dots, u_n)$ in both cases when $(u_n)_n$ is an arithmetic progression or when it is a quadratic progression. In the case of arithmetic progressions, Hong and Feng [7] and Hong and Yang [8] obtained some improvements of Farhi's lower bounds.

Among the arithmetic progressions, the sequences of consecutive integers are the most well-known with regards the properties of their least common multiple. In [4], Farhi introduced the arithmetic function $g_k : \mathbb{N}^* \to \mathbb{N}^*$ $(k \in \mathbb{N})$ which is defined by:

$$g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}(n, n+1, \dots, n+k)} \qquad (\forall n \in \mathbb{N}^*).$$

Farhi proved that the sequence $(g_k)_{k \in \mathbb{N}}$ satisfies the recursive relation:

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \qquad (\forall k, n \in \mathbb{N}^*). \tag{1}$$

Then, using this relation, he deduced (by induction on k) that g_k $(k \in \mathbb{N})$ is periodic and $k!$ is a period of g_k. A natural open problem raised in [4] consists to determine the exact period (i.e., the smallest positive period) of g_k.

For the following, let P_k denote the exact period of g_k. So, Farhi's result amounts that P_k divides $k!$ for all $k \in \mathbb{N}$. Very recently, Hong and Yang have shown that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$. This improves Farhi's result but it doesn't solve the raised problem of determining the P_k's. In their paper [8], Hong and Yang have also conjectured that P_k is a multiple of $\frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1}$ for all non-negative integer k. According to the property that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$ $(\forall k \in \mathbb{N})$, this conjecture implies that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds at least when $(k+1)$ is prime.

In this paper, we first prove the conjecture of Hong and Yang and then we give the exact value of P_k $(\forall k \in \mathbb{N})$. As a corollary, we show that P_k is equal to the part of $\mathrm{lcm}(1, 2, \dots, k)$ not divisible by some prime and that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds for an infinitely many $k \in \mathbb{N}$ for which $(k+1)$ is not prime.

2 Proof of the conjecture of Hong and Yang

We begin by extending the functions g_k ($k \in \mathbb{N}$) to $\mathbb{Z}$ as follows:
• We define $g_0 : \mathbb{Z} \rightarrow \mathbb{N}^*$ by $g_0(n) = 1$, $\forall n \in \mathbb{Z}$.
• If, for some $k \geq 1$, g_{k-1} is defined, then we define g_k by the relation:

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \qquad (\forall n \in \mathbb{Z}). \tag{1'}$$

Those extensions are easily seen to be periodic and to have the same period as their restriction to $\mathbb{N}^*$. The following proposition plays a vital role in what follows:

Proposition 2.1 *For any $k \in \mathbb{N}$,we have $g_k(0) = k!$.*

Proof. This follows by induction on k with using the relation $(1')$. ■

We now arrive at the theorem implying the conjecture of Hong and Yang.

Theorem 2.2 *For all $k \in \mathbb{N}$, we have:*

$$P_k = \frac{\operatorname{lcm}(1,2,\dots,k+1)}{k+1}.\gcd\left(P_k+k+1, \operatorname{lcm}(P_k+1, P_k+2, \dots, P_k+k)\right).$$

The proof of this theorem needs the following lemma:

Lemma 2.3 *For all $k \in \mathbb{N}$, we have:*

$$\operatorname{lcm}(P_k, P_k+1, \dots, P_k+k) = \operatorname{lcm}(P_k+1, P_k+2, \dots, P_k+k).$$

Proof of the Lemma. Let $k \in \mathbb{N}$ fixed. The required equality of the lemma is clearly equivalent to say that P_k divides $\operatorname{lcm}(P_k+1, P_k+2, \dots, P_k+k)$. This amounts to showing that for any prime number p:

$$v_p(P_k) \leq v_p\left(\operatorname{lcm}(P_k+1, \dots, P_k+k)\right) = \max_{1 \leq i \leq k} v_p(P_k+i). \tag{2}$$

So it remains to show (2). Let p be a prime number. Because P_k divides $\operatorname{lcm}(1,2,\dots,k)$ (according to the result of Hong and Yang [8]), we have $v_p(P_k) \leq v_p(\operatorname{lcm}(1,2,\dots,k))$, that is $v_p(P_k) \leq \max_{1 \leq i \leq k} v_p(i)$. So there exists $i_0 \in \{1,2,\dots,k\}$ such that $v_p(P_k) \leq v_p(i_0)$. It follows, according to the elementary properties of the p-adic valuation, that we have:

$$v_p(P_k) = \min\left(v_p(P_k), v_p(i_0)\right) \leq v_p(P_k+i_0) \leq \max_{1 \leq i \leq k} v_p(P_k+i),$$

which confirms (2) and completes this proof. ■

Proof of Theorem 2.2. Let $k \in \mathbb{N}$ fixed. The main idea of the proof is to calculate in two different ways the quotient $\frac{g_k(P_k)}{g_k(P_k+1)}$ and then to compare the obtained results. On one hand, we have from the definition of the function g_k:

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{P_k(P_k+1)\dots(P_k+k)}{\mathrm{lcm}(P_k, P_k+1, \dots, P_k+k)} \Big/ \frac{(P_k+1)(P_k+2)\dots(P_k+k+1)}{\mathrm{lcm}(P_k+1, P_k+2, \dots, P_k+k+1)}$$
$$= P_k \frac{\mathrm{lcm}(P_k+1, P_k+2, \dots, P_k+k+1)}{(P_k+k+1)\mathrm{lcm}(P_k, P_k+1, \dots, P_k+k)} \tag{3}$$

Next, using Lemma 2.3 and the well-known formula "$ab = \mathrm{lcm}(a,b)\mathrm{gcd}(a,b)$ $(\forall a, b \in \mathbb{N}^*)$", we have:

$$(P_k+k+1)\mathrm{lcm}(P_k, P_k+1, \dots, P_k+k) = (P_k+k+1)\mathrm{lcm}(P_k+1, P_k+2, \dots, P_k+k)$$
$$= \mathrm{lcm}\left(P_k+k+1, \mathrm{lcm}(P_k+1, \dots, P_k+k)\right)$$
$$\times \mathrm{gcd}\left(P_k+k+1, \mathrm{lcm}(P_k+1, \dots, P_k+k)\right)$$
$$= \mathrm{lcm}(P_k+1, P_k+2, \dots, P_k+k+1)\mathrm{gcd}\left(P_k+k+1, \mathrm{lcm}(P_k+1, \dots, P_k+k)\right).$$

By substituting this into (3), we obtain:

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{P_k}{\mathrm{gcd}\left(P_k+k+1, \mathrm{lcm}(P_k+1, \dots, P_k+k)\right)}. \tag{4}$$

On other hand, according to Proposition 2.1 and to the definition of P_k, we have:

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{k!}{g_k(1)} = \frac{\mathrm{lcm}(1, 2, \dots, k+1)}{k+1}. \tag{5}$$

Finally, by comparing (4) and (5), we get:

$$P_k = \frac{\mathrm{lcm}(1, 2, \dots, k+1)}{k+1}\mathrm{gcd}\left(P_k+k+1, \mathrm{lcm}(P_k+1, P_k+2, \dots, P_k+k)\right),$$

as required. The proof is complete. ■

From Theorem 2.2, we derive the following interesting corollary, which confirms the conjecture of Hong and Yang [8].

Corollary 2.4 *For all $k \in \mathbb{N}$, the exact period P_k of g_k is a multiple of the positive integer $\frac{\mathrm{lcm}(1,2,\dots,k,k+1)}{k+1}$. In addition, for all $k \in \mathbb{N}$ for which $(k+1)$ is prime, we have precisely $P_k = \mathrm{lcm}(1, 2, \dots, k)$.*

Proof. The first part of the corollary immediately follows from Theorem 2.2. Furthermore, we remark that if k is a natural number such that $(k+1)$ is prime, then we have $\frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1} = \mathrm{lcm}(1, 2, \dots, k)$. So, P_k is both a multiple and a divisor of $\mathrm{lcm}(1, 2, \dots, k)$. Hence $P_k = \mathrm{lcm}(1, 2, \dots, k)$. This finishes the proof of the corollary. ■

Now, we exploit the identity of Theorem 2.2 in order to obtain the p-adic valuation of P_k $(k \in \mathbb{N})$ for most prime numbers p.

Theorem 2.5 *Let $k \geq 2$ be an integer and $p \in [1, k]$ be a prime number satisfying:*

$$v_p(k+1) < \max_{1 \leq i \leq k} v_p(i). \tag{6}$$

Then, we have:

$$v_p(P_k) = \max_{1 \leq i \leq k} v_p(i).$$

Proof. The identity of Theorem 2.2 implies the following equality:

$$v_p(P_k) = \max_{1 \leq i \leq k+1}(v_p(i)) - v_p(k+1) + \min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k}(v_p(P_k + i))\right\}. \tag{7}$$

Now, using the hypothesis (6) of the theorem, we have:

$$\max_{1 \leq i \leq k+1}(v_p(i)) = \max_{1 \leq i \leq k}(v_p(i)) \tag{8}$$

and

$$\max_{1 \leq i \leq k+1}(v_p(i)) - v_p(k+1) > 0.$$

According to (7), this last inequality implies that:

$$\min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k} v_p(P_k + i)\right\} < v_p(P_k). \tag{9}$$

Let $i_0 \in \{1, 2, \ldots, k\}$ such that $\max_{1 \leq i \leq k} v_p(i) = v_p(i_0)$. Since P_k divides $\mathrm{lcm}(1, 2, \ldots, k)$, we have $v_p(P_k) \leq v_p(i_0)$, which implies that $v_p(P_k + i_0) \geq \min(v_p(P_k), v_p(i_0)) = v_p(P_k)$. Thus $\max_{1 \leq i \leq k} v_p(P_k + i) \geq v_p(P_k)$. It follows from (9) that

$$\min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k} v_p(P_k + i)\right\} = v_p(P_k + k + 1) < v_p(P_k). \tag{10}$$

So, we have

$$\min\left(v_p(P_k), v_p(k+1)\right) \leq v_p(P_k + k + 1) < v_p(P_k),$$

which implies that

$$v_p(k+1) < v_p(P_k)$$

and then, that

$$v_p(P_k + k + 1) = \min\left(v_p(P_k), v_p(k+1)\right) = v_p(k+1).$$

According to (10), it follows that

$$\min\left\{v_p(P_k+k+1), \max_{1\le i\le k} v_p(P_k+i)\right\} = v_p(k+1). \tag{11}$$

By substituting (8) and (11) into (7), we finally get:

$$v_p(P_k) = \max_{1\le i\le k} v_p(i),$$

as required. The theorem is proved. ■

Using Theorem 2.5, we can find infinitely many natural numbers k so that $(k+1)$ is not prime and the equality $P_k = \operatorname{lcm}(1, 2, \ldots, k)$ holds. The following corollary gives concrete examples for such numbers k.

Corollary 2.6 *If k is an integer having the form $k = 6^r - 1$ $(r \in \mathbb{N}, r \ge 2)$, then we have*

$$P_k = \operatorname{lcm}(1, 2, \ldots, k).$$

Consequently, there are an infinitely many $k \in \mathbb{N}$ for which $(k+1)$ is not prime and the equality $P_k = \operatorname{lcm}(1, 2, \ldots, k)$ holds.

Proof. Let $r \ge 2$ be an integer and $k = 6^r - 1$. We have $v_2(k+1) = v_2(6^r) = r$ while $\max_{1\le i\le k} v_2(i) \ge r+1$ (since $k \ge 2^{r+1}$). Thus $v_2(k+1) < \max_{1\le i\le k} v_2(i)$. Similarly, we have $v_3(k+1) = v_3(6^r) = r$ while $\max_{1\le i\le k} v_3(i) \ge r+1$ (since $k \ge 3^{r+1}$). Thus $v_3(k+1) < \max_{1\le i\le k} v_3(i)$.
Finally, for any prime $p \in [5, k]$, we clearly have $v_p(k+1) = v_p(6^r) = 0$ and $\max_{1\le i\le k} v_p(i) \ge 1$. Hence $v_p(k+1) < \max_{1\le i\le k} v_p(i)$.
This shows that the hypothesis of Theorem 2.5 is satisfied for any prime number p. Consequently, we have for any prime p: $v_p(P_k) = \max_{1\le i\le k} v_p(i) = v_p(\operatorname{lcm}(1, 2, \ldots, k))$. Hence $P_k = \operatorname{lcm}(1, 2, \ldots, k)$, as required. ■

3 Determination of the exact value of P_k

Notice that Theorem 2.5 successfully computes the value of $v_p(P_k)$ for almost all primes p (in fact we will prove in Proposition 3.3 that Theorem 2.5 fails to provide this value for at most one prime). In order to evaluate P_k, all we have left to do is compute $v_p(P_k)$ for primes p so that $v_p(k+1) \ge \max_{1\le i\le k} v_p(i)$. In particular we will prove:

Lemma 3.1 *Let $k \in \mathbb{N}$. If $v_p(k+1) \ge \max_{1\le i\le k} v_p(i)$, then $v_p(P_k) = 0$.*

From which the following result is immediate:

Theorem 3.2 *We have for all $k \in \mathbb{N}$:*

$$P_k = \prod_{p \text{ prime},\ p \le k} p^{\begin{cases} 0 & \text{if } v_p(k+1) \ge \max_{1 \le i \le k} v_p(i) \\ \max_{1 \le i \le k} v_p(i) & \text{else} \end{cases}}.$$

In order to prove this result, we will need to look into some of the more detailed divisibility properties of $g_k(n)$. In this spirit we make the following definitions:

Let $S_{n,k} = \{n, n+1, n+2, \ldots, n+k\}$ be the set of integers in the range $[n, n+k]$.

For a prime number p, let $g_{p,k}(n) := v_p(g_k(n))$. Let $P_{p,k}$ be the exact period of $g_{p,k}$. Since a positive integer is uniquely determined by the number of times each prime divides it, $P_k = \mathrm{lcm}_{p \text{ prime}}(P_{p,k})$.

Now note that

$$\begin{aligned} g_{p,k}(n) &= \sum_{m \in S_{n,k}} v_p(m) - \max_{m \in S_{n,k}} v_p(m) \\ &= \sum_{e>0, m \in S_{n,k}} (1 \text{ if } p^e | m) - \sum_{e>0} (1 \text{ if } p^e \text{ divides some } m \in S_{n,k}) \\ &= \sum_{e>0} \max(0, \#\{m \in S_{n,k} : p^e | m\} - 1). \end{aligned}$$

Let $e_{p,k} = \lfloor \log_p(k) \rfloor = \max_{1 \le i \le k} v_p(i)$ be the largest exponent of a power of p that is at most k. Clearly there is at most one element of $S_{n,k}$ divisible by p^e if $e > e_{p,k}$, therefore terms in the above sum with $e > e_{p,k}$ are all 0. Furthermore, for each $e \le e_{p,k}$, at least one element of $S_{p,k}$ is divisible by p^e. Hence we have that

$$g_{p,k}(n) = \sum_{e=1}^{e_{p,k}} \left(\#\{m \in S_{n,k} : p^e | m\} - 1 \right). \tag{12}$$

Note that each term on the right hand side of (12) is periodic in n with period $p^{e_{p,k}}$ since the condition $p^e | (n+m)$ for fixed m is periodic with period p^e. Therefore $P_{p,k} | p^{e_{p,k}}$. Note that this implies that the $P_{p,k}$ for different p are relatively prime, and hence we have that

$$P_k = \prod_{p \text{ prime},\ p \le k} P_{p,k}.$$

We are now prepared to prove our main result

Proof of Lemma 3.1. Suppose that $v_p(k+1) \geq e_{p,k}$. It clearly suffices to show that $v_p(P_{q,k}) = 0$ for each prime q. For $q \neq p$ this follows immediately from the result that $P_{q,k} | q^{e_{q,k}}$. Now we consider the case $q = p$.

For each $e \in \{1, \ldots, e_{p,k}\}$, since $p^e | k+1$, it is clear that $\#\{m \in S_{n,k} : p^e | m\} = \frac{k+1}{p^e}$, which implies (according to (12)) that $g_{k,n}$ is independent of n. Consequently, we have $P_{p,k} = 1$, and hence $v_p(P_{p,k}) = 0$, thus completing our proof. ■

Note that a slightly more complicated argument allows one to use this technique to provide an alternate proof of Theorem 2.5.

We can also show that the result in Theorem 3.2 says that P_k is basically $\mathrm{lcm}(1, 2, \ldots, k)$.

Proposition 3.3 *There is at most one prime p so that $v_p(k+1) \geq e_{p,k}$. In particular, by Theorem 3.2, P_k is either $\mathrm{lcm}(1, 2, \ldots, k)$, or $\frac{\mathrm{lcm}(1,2,\ldots,k)}{p^{e_{p,k}}}$ for some prime p.*

Proof. Suppose that for two distinct primes, $p, q \leq k$ that $v_p(k+1) \geq e_{p,k}$, and $v_q(k+1) \geq e_{q,k}$. Then

$$k+1 \geq p^{v_p(k+1)} q^{v_q(k+1)} \geq p^{e_{p,k}} q^{e_{q,k}} > \min\left(p^{e_{p,k}}, q^{e_{q,k}}\right)^2 = \min\left(p^{2e_{p,k}}, q^{2e_{q,k}}\right).$$

But this would imply that either $k \geq p^{2e_{p,k}}$ or that $k \geq q^{2e_{q,k}}$ thus violating the definition of either $e_{p,k}$ or $e_{q,k}$. ■

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