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Nontrivial lower bounds for the least common multiple of some finite sequences of integers

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Abstract

We present here a method which allows to derive a nontrivial lower bounds for the least common multiple of some finite sequences of integers. We obtain efficient lower bounds (which in a way are optimal) for the arithmetic progressions and lower bounds less efficient (but nontrivial) for quadratic sequences whose general term has the form $u_n = an(n+t)+b$ with $(a,t,b) \in \mathbb{Z}^3, a \geq 5, t \geq 0, \mathrm{gcd(a,b)} = 1$. From this, we deduce for instance the lower bound: $\mathrm{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\} \geq 0.32(1.442)^n$ (for all $n \geq 1$).

In the last part of this article, we study the integer $\mathrm{lcm}(n, n+1, \dots, n+k)$ $(k \in \mathbb{N}, n \in \mathbb{N} \setminus \{0\})$. We show that it has a divisor $d_{n,k}$ simple in its dependence on n and k, and a multiple $m_{n,k}$ also simple in its dependence on n. In addition, we prove that both equalities: $\mathrm{lcm}(n, n+1, \dots, n+k) = d_{n,k}$ and $\mathrm{lcm}(n, n+1, \dots, n+k) = m_{n,k}$ hold for an infinitely many pairs (n,k).

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1 Introduction and notations

In this article, $[x]$ denotes the integer part of a given real number x. Further, we say that a real x is a multiple of a non-zero real y if the ratio x/y is an integer.

The prime numbers theorem (see e.g. [2]) shows that $\lim_{n \to +\infty} \frac{\log \mathrm{lcm}\{1,\dots,n\}}{n} = 1$. This is equivalent to the following statement:

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \,/\, \forall n \geq N : \quad (e-\varepsilon)^n \ \leq \ \mathrm{lcm}\{1,\dots,n\} \ \leq \ (e+\varepsilon)^n.$$

Concerning the effective estimates of the numbers $\mathrm{lcm}\{1, \dots, n\}$ $(n \geq 1)$, one has among others, two main results. The first one is by Hanson [1] which shows (by using the development of the number 1 in Sylvester series) that $\mathrm{lcm}\{1, \dots, n\} \leq 3^n$ for all $n \geq 1$. The second one is by Nair [3] which proves (simply by exploiting the integral $\int_0^1 x^n(1-x)^n dx$) that one has $\mathrm{lcm}\{1, \dots, n\} \geq 2^n$ for all $n \geq 7$.

In this, we present a method which allows to find a nontrivial lower bounds for the least common multiple of n consecutive terms $(n \in \mathbb{N}^*)$ of some sequences of integers. We obtain efficient lower bounds (which in a way are optimal) for the arithmetical progressions (see Theorem 5). Besides, we also obtain less efficient lower bounds (but nontrivial) for the quadratic sequences whose general term has the form: $u_n = an(n+t) + b$ with $(a, t, b) \in \mathbb{Z}^3$, $a \geq 5, t \geq 0, \gcd(a, b) = 1$ (see Corollary 10).

Our method is based on the use of some identities related to the sequences which we study. More precisely, let $(\alpha_i)_{i \in I}$ be a given finite sequence of nonzero integers. We seek an identity of type $\sum_{i \in I} \frac{1}{\alpha_i \beta_i} = \frac{1}{\gamma}$ where β_i $(i \in I)$ and γ are nonzero integers. If $\mathrm{lcm}\{\beta_i, i \in I\}$ is bounded (say by a real constant $R > 0$), one concludes that $\mathrm{lcm}\{\alpha_i, i \in I\} \geq \frac{\gamma}{R}$ (see Lemma 1). It remains to check whether this later estimate is nontrivial or not.

However, the point is that looking for identities of the above types is not easy. Theorem 2 stems from concrete and interesting example of such identities. Though, it is not likewise that we can find other nontrivial applications, than the ones presented here, for that specific example. In order to have nontrivial lower bounds of least common multiple for other families of finite sequences, it could be necessary to seek for new identities related to those sequences.

In the last part of this article, we study the least common multiple of some number of consecutive integers, larger than a given positive integer. In Theorem 11, we show that the integer $\mathrm{lcm}\{n, n+1, \dots, n+k\}$ $(n \in \mathbb{N}^*, k \in \mathbb{N})$ has a divisor $d_{n,k}$ simple in its dependence on n and k and a multiple $m_{n,k}$ simple in its dependence on n. In addition, we prove that $d_{n,k}$ and $m_{n,k}$ are optimal in that sense that the equalities $\mathrm{lcm}\{n, \dots, n+k\} = d_{n,k}$ and $\mathrm{lcm}\{n, \dots, n+k\} = m_{n,k}$ hold for infinitely many pairs (n, k). More precisely, we show that both equalities are satisfied at least when (n, k) satisfies some congruence modulo $k!$ (see Theorem 12).

2 Results

2.1 Basic Results

Lemma 1 *Let $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ be two finite sequences of non-zero integers such that:*

$$\sum_{i\in I} \frac{1}{\alpha_i\beta_i} = \frac{1}{\gamma}$$

for some non-zero integer γ. Then, the integer $\mathrm{lcm}\{\alpha_i, i\in I\}.\mathrm{lcm}\{\beta_i, i\in I\}$ is a multiple of γ.

Theorem 2 *Let $(u_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of non-zero integers. Then, for any positive integer n, the integer:*

$$\mathrm{lcm}\left\{u_0,\dots,u_n\right\}.\mathrm{lcm}\left\{\prod_{0\le i\le n, i\ne j}(u_i-u_j)\ ;\ j=0,\dots,n\right\}$$

is a multiple of the integer $(u_0u_1\dots u_n)$.

2.2 Results about the arithmetic progressions

Theorem 3 *Let $(u_k)_{k\in\mathbb{N}}$ be a strictly increasing arithmetic progression of non-zero integers. Then, for any non-negative integer n, the integer $\mathrm{lcm}\{u_0,\dots,u_n\}$ is a multiple of the rational number:*

$$\frac{u_0\dots u_n}{n!\left(\gcd\{u_0,u_1\}\right)^n}.$$

Theorem 4 (Optimality of Theorem 3) *Let $(u_k)_{k\in\mathbb{N}}$ be a strictly increasing arithmetic progression of non-zero integers such that u_0 and u_1 are coprime. Then, for any positive integer n which satisfies:*

$$u_0u_n \equiv 0 \bmod (n!),$$

we have:

$$\mathrm{lcm}\{u_0,\dots,u_n\} = \frac{u_0\dots u_n}{n!}.$$

Theorem 5 *Let $(u_k)_{k\in\mathbb{N}}$ be an arithmetic progression of integers whose difference r and first term u_0 are positive and coprime. Then:*

1) *For any $n \in \mathbb{N}$, we have:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ u_0(r+1)^{n-1}.$$

Besides, if n is a multiple of $(r+1)$, we have:

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ u_0(r+1)^{n}.$$

2) *For any $n \in \mathbb{N}$, we have:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ r(r+1)^{n-1}.$$

3) *For any $n \in \mathbb{N}$, we have:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ \frac{n}{n+1} r \left\{(r+1)^{n-1} + (r-1)^{n-1}\right\}.$$

4) *For any $n \in \mathbb{N}$ satisfying $n \geq u_0 - \frac{3r+1}{2}$, we have:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ \frac{1}{\pi}\sqrt{r}(r+1)^{n-1+\frac{u_0}{r}}.$$

The following Conjecture improves the parts **1)** and **2)** of Theorem 5. Besides, the first part of this Theorem ensures its validity in the particular case where the integer n is a multiple of $(r+1)$.

Conjecture 6 *In the situation of Theorem 5, we have for any $n \in \mathbb{N}$:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ u_0(r+1)^{n}.$$

The two following Theorems study the optimality of the part **4)** of Theorem 5.

Theorem 7 *The coefficient $-\frac{3}{2}$ affected to r which appears in the condition "$n \geq u_0 - \frac{3r+1}{2}$" of the part* **4)** *of Theorem 5 is optimal.*

Theorem 8

1) *The optimal absolute constant C for which the assertion:*

> *"For any arithmetic sequence $(u_k)_k$ as in Theorem 5 and for any non-negative integer n satisfying $n \geq u_0 - \frac{3r+1}{2}$, we have:*
> $\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ C\sqrt{r}(r+1)^{n-1+\frac{u_0}{r}}$"

is true, satisfies:

$$\frac{1}{\pi} \ \leq \ C \ \leq \ \frac{3}{2}.$$

2) *More generally, given $n_0 \in \mathbb{N}$, the optimal constant $C(n_0)$ (depending uniquely on n_0) for which the assertion:*

> *"For any arithmetic sequence $(u_k)_k$ as in Theorem 5 and for any integer n satisfying $n \geq \max\{n_0, u_0 - \frac{3r+1}{2}\}$, we have:*
> $\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ C(n_0)\sqrt{r}(r+1)^{n-1+\frac{u_0}{r}}$"

is true, satisfies:

$$\frac{1}{\pi} \ \leq \ C(n_0) \ < \ 4(n_0+4)\sqrt{n_0+4}.$$

Comments:

i) The lower bound proposed by Conjecture 6 is optimal on the exponent n of $(r+1)$. Indeed, for any positive integer n and for any arithmetic progression $(u_k)_k$ as in Theorem 5, we obviously have:

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \leq \ u_0 u_1 \dots u_n \ \leq \ u_0 \left(\max\{u_0, n\}\right)^n (r+1)^n.$$

For any given positive real ε, we can choose two arbitrary positive integers u_0 and n and a positive integer r, which is coprime with u_0 and sufficiently large as to have $(r+1)^{\varepsilon} > \left(\max\{u_0, n\}\right)^n$. The arithmetic progression $(u_k)_k$, with first term u_0 and difference r, will then satisfy:

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ < \ u_0(r+1)^{n+\varepsilon}.$$

ii) A similar argument to that of the above part **i)** shows that the exponent $(n-1)$ of $(r+1)$ which appears in the lower bound of the part **2)** of Theorem 5 is optimal.

iii) For small values of n according to r, the lower bound of the part **3)** of Theorem 5 implies the one of the part **2)** of the same Theorem. More precisely, it can be checked that the necessary and sufficiently condition for the holding of this improvement is $r \geq \frac{n^{\frac{1}{n-1}}+1}{n^{\frac{1}{n-1}}-1}$, that is $n \leq f(r)$, where f is a real function which is equivalent to $\frac{1}{2} r \log r$ as r tends to infinite.

iv) Under the additional assumptions $7 \leq r \leq 2u_0$ and $n \geq u_0 - \frac{3r+1}{2}$ (resp. $r \leq 2u_0$ and $n \geq u_0 - \frac{3r+1}{2}$), the lower bound of the part **4)** of Theorem 5 implies the one of the part **1)** (resp. **2)**) of the same Theorem up to the multiplicative constant $\frac{2}{\pi}$ (resp. $\frac{1}{\pi}$).
(Notice that the function $x \mapsto \sqrt{x}(x+1)^{\frac{u_0}{x}}$ is decreasing on the interval $[7, 2u_0]$, then if $7 \leq r \leq 2u_0$, we have $\sqrt{r}(r+1)^{\frac{u_0}{r}} > 2u_0$).

v) Now, we check that if $r \leq \frac{2}{3}u_0$ and $n \geq u_0 - \frac{3r+1}{2}$, the lower bound of the part **4)** of Theorem 5 implies (up to a multiplicative constant) the one of Conjecture 6. Indeed, if $r \leq \frac{2}{3}u_0$ and $n \geq u_0 - \frac{3r+1}{2}$, the decrease of the function $x \mapsto \sqrt{x}(x+1)^{\frac{u_0}{x}-1}$ on the interval $[1, +\infty[$ implies: $\sqrt{r}(r+1)^{\frac{u_0}{r}-1} \geq \sqrt{\frac{2}{3}u_0}\left(\frac{2}{3}u_0+1\right)^{\frac{1}{2}} > \frac{2}{3}u_0$ which gives (by using the lower bound of the part **4)** of Theorem 5):

$$\mathrm{lcm}\{u_0, \dots, u_n\} \;\geq\; \frac{2}{3\pi}u_0(r+1)^n.$$

• More generally, for any given real $\xi \geq \frac{3}{2}$, if we suppose $r \leq \frac{1}{\xi}u_0$ and $n \geq u_0 - \frac{3r+1}{2}$ then the decrease of the function $x \mapsto \sqrt{x}(x+1)^{\frac{u_0}{x}-\xi+\frac{1}{2}}$ on the interval $[1, +\infty[$ implies: $\sqrt{r}(r+1)^{\frac{u_0}{r}-\xi+\frac{1}{2}} \geq \sqrt{\frac{u_0}{\xi}}\left(\frac{u_0}{\xi}+1\right)^{\frac{1}{2}} > \frac{u_0}{\xi}$ which gives (by using the lower bound of the part **4)** of Theorem 5):

$$\mathrm{lcm}\{u_0, \dots, u_n\} \;\geq\; \frac{1}{\pi\xi}u_0(r+1)^{n+\xi-\frac{3}{2}}.$$

Remark that if $\xi > \frac{3}{2}$, this lower bound is stronger than the one of Conjecture 6.

2.3 Results about the quadratic sequences

Theorem 9 *Let $\mathbf{u} = (u_k)_{k\in\mathbb{N}}$ be a sequence of integers whose general term has the form:*

$$u_k \;=\; ak(k+t)+b \qquad (\forall k \in \mathbb{N}),$$

with $(a, t, b) \in \mathbb{Z}^3$, $a \geq 1$, $t \geq 0$ and $\gcd\{a, b\} = 1$.
Also let m and n (with $m < n$) be two non-negative integers for which none of the terms u_k $(m \leq k \leq n)$ of $\mathbf{u}$ is zero. Then the integer $\mathrm{lcm}\{u_m, \dots, u_n\}$ is a multiple of the rational number:

$$A_{\mathbf{u}}(t, m, n) \;:=\; \begin{cases} 2\frac{u_0 \cdots u_n}{(2n)!} & \textit{if } (t, m) = (0, 0) \\ (2m+t-1)!\frac{u_m \cdots u_n}{(2n+t)!} & \textit{otherwise} \end{cases}.$$

Corollary 10 *Let $\mathbf{u} = (u_k)_{k\in\mathbb{N}}$ be a sequence of integers as in the above Theorem and n be a positive integer. Then, if the $(n+1)$ first terms $u_0, \dots, u_n$ of the sequence $\mathbf{u}$ are all non-zero, then we have:*

$$\mathrm{lcm}\{u_0, \dots, u_n\} \;\geq\; \begin{cases} 2b\left(\frac{a}{4}\right)^n & \textit{if } t = 0 \\ \frac{b}{t2^t}\left(\frac{a}{4}\right)^n & \textit{if } t \geq 1 \end{cases}.$$

Remark. It is clear that the lower bound of Corollary 10 is nontrivial only if $a \geq 5$. Such as it is, this corollary cannot thus give a nontrivial lower bound for the numbers $\operatorname{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\}$ $(n \geq 1)$. But we remark that if $r \geq 3$ is an integer, it gives a nontrivial lower bound for the last common multiple of consecutive terms of the sequence $(r^2n^2+1)_{n\geq 1}$ which is a subsequence of $(n^2+1)_n$. So we can first obviously bound from below $\operatorname{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\}$ by $\operatorname{lcm}\{r^2+1, r^2 2^2+1, \dots, r^2k^2+1\}$ (with $k := [\frac{n}{r}]$), then use Corollary 10 to bound from below this new quantity. We obtain in this way:

$$\operatorname{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\} \;\geq\; 2\left(\frac{r^2}{4}\right)^k \;>\; 2\left(\frac{r^2}{4}\right)^{\frac{n}{r}-1} \;=\; \frac{8}{r^2}\left\{\left(\frac{r}{2}\right)^{\frac{2}{r}}\right\}^n.$$

This gives (for any choice of $r \geq 3$) a nontrivial lower bound for the numbers $\operatorname{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\}$ $(n \geq 1)$. We easily verify that the optimal lower bound corresponds to $r = 5$, that is:

$$\operatorname{lcm}\{1^2+1, 2^2+1, \dots, n^2+1\} \;\geq\; 0,32(1,442)^n \qquad (\forall n \geq 1).$$

2.4 Results about the least common multiple of a finite number of consecutive integers

The following Theorem is an immediate consequence of Theorems 3 and 4.

Theorem 11 *For any non-negative integer k and any positive integer n, the integer $\operatorname{lcm}\{n, n+1, \dots, n+k\}$ is a multiple of the integer $n\binom{n+k}{k}$.*
Further, if the congruence $n(n+k) \equiv 0 \bmod (k!)$ is satisfied, then we have precisely:

$$\operatorname{lcm}\{n, n+1, \dots, n+k\} \;=\; n\binom{n+k}{k}.$$

The following result is independent of all the results previously quoted. It gives a multiple $m_{n,k}$ of the integer $\operatorname{lcm}\{n, n+1, \dots, n+k\}$ $(k \in \mathbb{N}, n \in \mathbb{N}^*)$ which is optimal and simple in its dependance on n.

Theorem 12 *For any non-negative integer k and any positive integer n, the integer $\operatorname{lcm}\{n, n+1, \dots, n+k\}$ divides the integer $n\binom{n+k}{k}\operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}$.*
Further, if the congruence $n+k+1 \equiv 0 \bmod (k!)$ is satisfied, then we have precisely:

$$\operatorname{lcm}\{n, n+1, \dots, n+k\} \;=\; n\binom{n+k}{k}\operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}.$$

3 Proofs

Proof of Lemma 1. In the situation of Lemma 1, we have:

$$\begin{aligned}\frac{\mathrm{lcm}\{\alpha_i, i \in I\}.\mathrm{lcm}\{\beta_i, i \in I\}}{\gamma} &= \mathrm{lcm}\{\alpha_i, i \in I\}.\mathrm{lcm}\{\beta_i, i \in I\} \sum_{j \in I} \frac{1}{\alpha_j \beta_j} \\ &= \sum_{j \in I} \frac{\mathrm{lcm}\{\alpha_i, i \in I\}}{\alpha_j} . \frac{\mathrm{lcm}\{\beta_i, i \in I\}}{\beta_j} .\end{aligned}$$

This last sum is clearly an integer because for any $j \in I$, the two numbers $\frac{\mathrm{lcm}\{\alpha_i, i \in I\}}{\alpha_j}$ and $\frac{\mathrm{lcm}\{\beta_i, i \in I\}}{\beta_j}$ are integers. Lemma 1 follows. ■

Proof of Theorem 2. Theorem 2 follows by applying Lemma 1 to the identity:

$$\sum_{j=0}^{n} \frac{1}{u_j} . \frac{1}{\prod_{0 \leq i \leq n, i \neq j} (u_i - u_j)} = \frac{1}{u_0 u_1 \dots u_n},$$

which we obtain by taking $x = 0$ in the decomposition to simple elements of the rational fraction $x \mapsto \frac{1}{(x+u_0)(x+u_1)\dots(x+u_n)}$. ■

Proof of Theorem 3. By replacing if necessary the sequence $(u_n)_n$ by the sequence with general term $v_n := \frac{u_n}{\gcd\{u_0, u_1\}}$ ($\forall n \in \mathbb{N}$), we may assume that u_0 and u_1 are coprime. Under this hypothesis, we have to show that the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number $\frac{u_0 \dots u_n}{n!}$ (for any $n \in \mathbb{N}$).

Let n be a fixed non-negative integer. From Theorem 2, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number

$$\frac{u_0 \dots u_n}{\mathrm{lcm}\left\{ \prod_{0 \leq i \leq n, i \neq j} (u_i - u_j) \ ; \ 0 \leq j \leq n \right\}}.$$

Let r denotes the difference of the arithmetic sequence $(u_k)_k$. We have for any $(i, j) \in \mathbb{N}^2$: $u_i - u_j = (i - j)r$, then for any $j \in \{0, \dots, n\}$:

$$\begin{aligned}\prod_{0 \leq i \leq n, i \neq j} (u_i - u_j) &= \prod_{0 \leq i \leq n, i \neq j} (i - j)r \\ &= r^n \left\{(-j)(1-j)(2-j)\dots(-1)\right\}.\left\{1.2\dots(n-j)\right\} \\ &= r^n (-1)^j j!(n-j)!.\end{aligned}$$

Hence:

$$\begin{aligned}\mathrm{lcm}\left\{\prod_{0\leq i\leq n, i\neq j}(u_i-u_j)\ ;\ 0\leq j\leq n\right\} &= \mathrm{lcm}\left\{r^n(-1)^j j!(n-j)!\ ;\ 0\leq j\leq n\right\}\\ &= r^n \mathrm{lcm}\left\{j!(n-j)!\ ;\ 0\leq j\leq n\right\}\\ &= r^n n!\end{aligned}$$

(because each integer $j!(n-j)!$ divides $n!$ and for $j=0$ or n, we have $j!(n-j)!=n!$).
Thus the integer $\mathrm{lcm}\{u_0,\dots,u_n\}$ is a multiple of the rational number $\frac{u_0\dots u_n}{r^n n!}$. But our hypothesis "$u_0$ coprime with u_1" implies that r is coprime with all terms of the sequence $(u_k)_k$, which implies that r^n is coprime with the product $u_0\dots u_n$. By the Gauss lemma, we finally conclude that the integer $\mathrm{lcm}\{u_0,\dots,u_n\}$ is a multiple of the rational number $\frac{u_0\dots u_n}{n!}$ as required. ■

Proof of Theorem 4. We need the following preliminary Lemma:

Lemma. *Let n be a positive integer and x and y be two integers satisfying: $x-y\equiv 0 \bmod(n)$ and $xy\equiv 0 \bmod(n!)$. Then x and y are multiples of n.*

Proof. We distinguish the following four cases:
• If $n=1$: In this case, the result of Lemma is trivial.
• If n is prime: In this case, since $x^2=x(x-y)+xy$, we have $x^2\equiv 0 \bmod(n)$, but since n is supposed prime, we conclude that $x\equiv 0 \bmod(n)$ and then that $y=x-(x-y)\equiv 0 \bmod(n)$.
• If $n=4$: In this case, we have $x-y\equiv 0 \bmod(4)$ and $xy\equiv 0 \bmod(24)$ and we have to show that x and y are multiples of 4. Let us argue by contradiction. Then, since $x\equiv y \bmod(4)$, we have:
— Either $x\equiv y\equiv 1,3 \bmod(4)$ which implies $xy\equiv 1 \bmod(4)$ and contradicts the congruence $xy\equiv 0 \bmod(24)$.
— Or $x\equiv y\equiv 2 \bmod(4)$ which implies $xy\equiv 4 \bmod(8)$ and contradicts the congruence $xy\equiv 0 \bmod(24)$ again.
Thus the Lemma holds for $n=4$.
• If $n\geq 5$ and n is not prime: In this case, it is easy to see that the integer $(n-1)!$ is a multiple of n, so that the integer $n!$ is a multiple of n^2. We thus have $x-y\equiv 0 \bmod(n)$ and $xy\equiv 0 \bmod(n^2)$.

Let us argue by contradiction. Suppose that one at least of the two integers x and y is not a multiple of n. To fix the ideas, suppose for instance that $x\not\equiv 0 \bmod(n)$. Then, there exists a prime number p dividing n such that $v_p(x)<v_p(n)$. But since $xy\equiv 0 \bmod(n^2)$, we have $v_p(xy)\geq v_p(n^2)$, that is $v_p(x)+v_p(y)\geq 2v_p(n)$. This implies that $v_p(y)\geq 2v_p(n)-v_p(x)>v_p(x)$ (because $v_p(x)<v_p(n)$). Thus, the p-adic valuations of the integers x and y are distinct. Then we have: $v_p(x-y)=\min(v_p(x),v_p(y))=v_p(x)<v_p(n)$, which

contradicts the fact that $(x - y)$ is a multiple of n. The Lemma is proved.

Return to the proof of Theorem 4:

The case $n = 1$ is trivial. Next, we assume that $n \geq 2$. From Theorem 3, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number $\frac{u_0 \dots u_n}{n!}$. To prove Theorem 4, it remains to prove that $\frac{u_0 \dots u_n}{n!}$ is also a multiple of $\mathrm{lcm}\{u_0, \dots, u_n\}$, which means that $\frac{u_0 \dots u_n}{n!}$ is a multiple of each of integers $u_0, \dots, u_n$. Since $u_0 u_n$ is assumed a multiple of $n!$, the number $\frac{u_0 \dots u_n}{n!}$ is obviously a multiple of each of integers $u_1, \dots, u_{n-1}$. To conclude, it only remains to prove that this same number $\frac{u_0 \dots u_n}{n!}$ is a multiple of u_0 and u_n, which is equivalent to prove that the two integers $u_1 \dots u_n$ and $u_0 \dots u_{n-1}$ are multiples of $n!$. We first prove that u_0 and u_n are multiples of n. Denoting r the difference of the arithmetic sequence $(u_k)_k$, we have $u_n - u_0 = rn \equiv 0 \mod (n)$ and $u_0 u_n \equiv 0 \mod (n!)$ (by hypothesis). This implies (from the above Lemma) that u_0 and u_n effectively are multiples of n.

We now prove that the two integers $u_1 \dots u_n$ and $u_0 \dots u_{n-1}$ are multiples of $n!$. For any $1 \leq k \leq n-1$, we have: $u_k = u_0 + kr \equiv kr \mod (u_0)$, then:

$$u_1 \dots u_{n-1} \ \equiv \ (1.r)(2.r) \dots ((n-1).r) \mod (u_0) \ \equiv \ (n-1)! r^{n-1} \mod (u_0).$$

It follows that:

$$u_1 \dots u_{n-1} u_n \ \equiv \ (n-1)! u_n r^{n-1} \mod (u_0 u_n).$$

Since u_n is a multiple of n and (by hypothesis) $u_0 u_n$ is a multiple of $n!$, the last congruence implies that $u_1 \dots u_{n-1} u_n$ is a multiple of $n!$.

Similarly, for any $1 \leq k \leq n-1$, we have: $u_{n-k} = u_n - kr \equiv -kr \mod (u_n)$, then:

$$u_{n-1} \dots u_1 \ \equiv \ (-(n-1).r) \dots (-1.r) \mod (u_n) \ \equiv \ (-1)^{n-1} (n-1)! r^{n-1} \mod (u_n).$$

It follows that:

$$u_0 u_1 \dots u_{n-1} \ \equiv \ (-1)^{n-1} (n-1)! u_0 r^{n-1} \mod (u_0 u_n).$$

Since u_0 is a multiple of n and (by hypothesis) $u_0 u_n$ is a multiple of $n!$, the last congruence implies that $u_0 \dots u_{n-1}$ is also a multiple of $n!$. This completes the proof of Theorem 4. ■

Proof of Theorem 5. For any integer $k \in \{0, \dots, n\}$, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is obviously a multiple of the integer $\mathrm{lcm}\{u_k, \dots, u_n\}$ and from Theorem 3, this last integer is a multiple of the rational number $\frac{u_k \dots u_n}{(n-k)!}$. It follows that for any $k \in \{0, \dots, n\}$, we have:

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ \frac{u_k \dots u_n}{(n-k)!}. \tag{1}$$

The idea consists in choosing k as a function of n, r and u_0 in order to optimize the lower bound (1), that is to make the quantity $\frac{u_k \dots u_n}{(n-k)!}$ maximal.

Let $(v_k)_{0 \leq k \leq n}$ denotes the finite sequence of general term: $v_k := \frac{u_k \dots u_n}{(n-k)!}$. We have the following intermediate Lemma:

Lemma. *The sequence $(v_k)_{0 \leq k \leq n}$ reaches its maximum value at*

$$k_0 := \max\left\{0, \left[\frac{n-u_0}{r+1}\right] + 1\right\}.$$

Proof. For any $k \in \{0, \dots, n-1\}$, we have: $\frac{v_{k+1}}{v_k} = \frac{u_{k+1} \dots u_n}{(n-k-1)!} / \frac{u_k \dots u_n}{(n-k)!} = \frac{n-k}{u_k} = \frac{n-k}{u_0+kr}$, hence:

$$v_{k+1} \geq v_k \iff \frac{n-k}{u_0+kr} \geq 1 \iff k \leq \frac{n-u_0}{r+1} \iff k \leq \left[\frac{n-u_0}{r+1}\right].$$

This permits us to determine the variations of the finite sequence $(v_k)_{0 \leq k \leq n}$ according to the position of n compared to u_0. If $n < u_0$, the sequence $(v_k)_{0 \leq k \leq n}$ is decreasing and it thus reaches its maximum value at $k = 0$. In the other case i.e $n \geq u_0$, the sequence $(v_k)_{0 \leq k \leq n}$ is increasing until the integer $\left[\frac{n-u_0}{r+1}\right] + 1$ then it decreases, so it reaches its maximum value at $k = \left[\frac{n-u_0}{r+1}\right] + 1$. The Lemma follows.

The following intermediary lemma gives an identity which permits to bound from bellow v_k by simple expressions (as function as u_0, r and n) for the particular values of k which are rather close to the integer k_0 of the above Lemma.

Lemma. *For any $k \in \{0, \dots, n\}$, we have:*

$$v_k = \frac{r^{n-k+1}}{\int_0^1 x^{k+\frac{u_0}{r}-1}(1-x)^{n-k}dx}. \tag{2}$$

Proof. For any $0 \leq k \leq n$, we have:

$$\begin{aligned} v_k := \frac{u_k \dots u_n}{(n-k)!} &= \frac{u_k\,(u_k+r)\dots(u_k+(n-k)r)}{(n-k)!} \\ &= r^{n-k+1}\frac{\frac{u_k}{r}\left(\frac{u_k}{r}+1\right)\dots\left(\frac{u_k}{r}+n-k\right)}{(n-k)!} \\ &= r^{n-k+1}\frac{\Gamma\left(\frac{u_k}{r}+n-k+1\right)}{\Gamma\left(\frac{u_k}{r}\right).\Gamma\left(n-k+1\right)} \\ &= \frac{r^{n-k+1}}{\beta\left(\frac{u_k}{r}, n-k+1\right)}, \end{aligned}$$

where Γ and β denote the Euler's functions. The identity (2) of Lemma follows from the well known integral formula of the β-function. The Lemma is proved.

Because of some technical difficulties concerning the lower bound of the right-hand side of (2) for $k = k_0$, we are led to bound from below this side for other values of k which are close to k_0. So, we obtain the lower bounds of the parts **1)** and **4)** of Theorem 5 by bounding from below v_k for $k = \left[\frac{n-1}{r+1} + 1\right]$ and for the nearest integer k to the real $\frac{n+r-u_0}{r+1}$ respectively. Further, we obtain the remaining parts **2)** and **3)** of Theorem 5 by another method which doesn't use the identity (2). We first prove the parts **1)** and **4)** of Theorem 5.

Proof of the part 1) of Theorem 5:

Let $k_1 := \left[\frac{n-1}{r+1} + 1\right]$. Using the identity (2), we are going to get a lower bound for v_{k_1} which depends on u_0, r and n. The integer k_1 satisfies $\frac{n-1}{r+1} < k_1 \leq \frac{n-1}{r+1} + 1 = \frac{n+r}{r+1}$. We thus have:

$$r^{n-k_1+1} \;\geq\; r^{\frac{(n-1)r}{r+1}+1} \tag{3}$$

and for any real $x \in [0,1]$:

$$x^{k_1+\frac{u_0}{r}-1}(1-x)^{n-k_1} \;\leq\; x^{\frac{n-1}{r+1}+\frac{u_0}{r}-1}(1-x)^{\frac{(n-1)r}{r+1}},$$

which gives:

$$\int_0^1 x^{k_1+\frac{u_0}{r}-1}(1-x)^{n-k_1}dx \;\leq\; \int_0^1 \{x(1-x)^r\}^{\frac{n-1}{r+1}} x^{\frac{u_0}{r}-1}dx. \tag{4}$$

By studying the function $x \mapsto x(1-x)^r$, we may show that for any real $x \in [0,1]$, we have: $x(1-x)^r \leq \frac{r^r}{(r+1)^{r+1}}$. Substituting this into the right-hand side of (4), we deduce that:

$$\int_0^1 x^{k_1+\frac{u_0}{r}-1}(1-x)^{n-k_1}dx \;\leq\; \frac{r^{\frac{(n-1)r}{r+1}}}{(r+1)^{n-1}}.\frac{r}{u_0} \tag{5}$$

By combining the two relations (3) and (5), we finally obtain:

$$\frac{r^{n-k_1+1}}{\int_0^1 x^{k_1+\frac{u_0}{r}-1}(1-x)^{n-k_1}dx} \;\geq\; u_0(r+1)^{n-1}.$$

Then the first lower bound of the part 1) of Theorem 5 follows from the relations (2) and (1).

If n is a multiple of $(r+1)$, the second lower bound of the part 1) of Theorem 5 follows by taking in the above proof instead of k_1 the integer $k = \frac{n}{r+1}$.

Proof of the part 4) of Theorem 5:

The particular case $n = 0$ of the part 4) of Theorem 5 follows from the fact that the function $x \mapsto \sqrt{x}(x+1)^{\frac{u_0}{x}-1}$ is decreasing on the interval $[1, +\infty[$. Next, we suppose that $n \geq 1$. The hypothesis $n \geq u_0 - \frac{3r+1}{2}$ means that the real $\frac{n+r-u_0}{r+1}$ is greater than or equal to $-\frac{1}{2}$. Since this same real $\frac{n+r-u_0}{r+1}$ is less than or equal to $n + \frac{1}{2}$ (because $n \geq 1$), then there exists an integer $k_2 \in \{0, \dots, n\}$ satisfying:

$$-\frac{1}{2} \;\leq\; k_2 - \frac{n+r-u_0}{r+1} \;\leq\; \frac{1}{2}.$$

It follows that:

$$r^{n-k_2+1} \;\geq\; r^{\frac{r(n-1)+u_0}{r+1}+\frac{1}{2}} \tag{6}$$

and that for any real $x \in]0,1[$:

$$\begin{aligned}
x^{k_2+\frac{u_0}{r}-1}(1-x)^{n-k_2} &\leq x^{\frac{r(n-1)+u_0}{r(r+1)}-\frac{1}{2}}(1-x)^{\frac{r(n-1)+u_0}{r+1}-\frac{1}{2}} \\
&= \{x(1-x)^r\}^{\frac{r(n-1)+u_0}{r(r+1)}} \frac{1}{\sqrt{x(1-x)}} \\
&\leq \left(\frac{r^r}{(r+1)^{(r+1)}}\right)^{\frac{r(n-1)+u_0}{r(r+1)}} \frac{1}{\sqrt{x(1-x)}}
\end{aligned}$$

(because $x(1-x)^r \leq \frac{r^r}{(r+1)^{r+1}}$ for any $x \in [0,1]$).
Consequently:

$$\int_0^1 x^{k_2+\frac{u_0}{r}-1}(1-x)^{n-k_2}dx \;\leq\; \left(\frac{r^r}{(r+1)^{(r+1)}}\right)^{\frac{r(n-1)+u_0}{r(r+1)}} \int_0^1 \frac{dx}{\sqrt{x(1-x)}}.$$

Since $\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi$, we deduce that:

$$\int_0^1 x^{k_2+\frac{u_0}{r}-1}(1-x)^{n-k_2}dx \;\leq\; \pi \frac{r^{\frac{r(n-1)+u_0}{r+1}}}{(r+1)^{n-1+\frac{u_0}{r}}}. \tag{7}$$

By combining the two relations (6) and (7), we finally obtain:

$$\frac{r^{n-k_2+1}}{\int_0^1 x^{k_2+\frac{u_0}{r}-1}(1-x)^{n-k_2}} \;\geq\; \frac{1}{\pi}\sqrt{r}(r+1)^{n-1+\frac{u_0}{r}}$$

and we conclude the lower bound of the part 4) of Theorem 5 by using the identity (2) and the lower bound (1).

We obtain the two remaining parts 2) and 3) of Theorem 5 by using the same idea which consists to bound from below $v_k = \frac{u_k \dots u_n}{(n-k)!}$ for some particular values of $k \in \{0, \dots, n\}$. The only difference with the last parts 1) and 4) proved above is that here such particular values are not explicit, we just show their existence by using the following Lemma:

Lemma. Let x be a real and n be a positive integer. Then:

1) there exists an integer k $(1 \leq k \leq n)$ such that:

$$k\binom{n}{k}x^{n-k+1} \ \geq \ x(x+1)^{n-1}.$$

2) There exists an odd integer ℓ $(1 \leq \ell \leq n)$ such that:

$$\ell\binom{n}{\ell}x^{n-\ell+1} \ \geq \ \frac{n}{n+1}x\left\{(x+1)^{n-1} + (x-1)^{n-1}\right\}.$$

Proof. The first part of Lemma follows from the identity:

$$\sum_{k=1}^{n} k\binom{n}{k}x^{n-k+1} \ = \ nx(x+1)^{n-1} \tag{8}$$

which can be proved by deriving with respect to u the binomial formula $\sum_{k=0}^{n}\binom{n}{k}u^k x^{n-k} = (u+x)^n$ and then by taking $u = 1$ in the obtained formula. The second part of Lemma follows from the identity:

$$\sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} k\binom{n}{k}x^{n-k+1} \ = \ \frac{1}{2}nx\left\{(x+1)^{n-1} + (x-1)^{n-1}\right\} \tag{9}$$

which follows from (8) by remarking that:

$$\sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} k\binom{n}{k}x^{n-k+1} \ = \ \frac{1}{2}\left\{\sum_{k=1}^{n} k\binom{n}{k}x^{n-k+1} + (-1)^n \sum_{k=1}^{n} k\binom{n}{k}(-x)^{n-k+1}\right\}.$$

The Lemma is proved.

Proof of the parts 2) and 3) of Theorem 5:

We have for any $k \in \{1, \dots, n\}$:

$$v_k := \frac{u_k \dots u_n}{(n-k)!} \ \geq \ \frac{(kr)((k+1)r)\dots(nr)}{(n-k)!} = k\binom{n}{k} r^{n-k+1}.$$

These lower bounds of v_k $(1 \leq k \leq n)$ implie (by using the above Lemma) that there exist an integer $k \in \{1, \dots, n\}$ and an odd integer $\ell \in \{1, \dots, n\}$ for which we have:

$$v_k \ \geq \ r(r+1)^{n-1} \ \text{ and } \ v_\ell \ \geq \ \frac{n}{n+1} r \left\{(r+1)^{n-1} + (r-1)^{n-1}\right\}.$$

We conclude by using Relation (1). This completes the proof of Theorem 5. ■

Proof of Theorem 7. Let us argue by contradiction. Then, we can find a rational number $\frac{a}{b} > \frac{3}{2}$ (with a, b are positive integers) for which we have for any arithmetic progression $(u_k)_k$ with positive difference r, satisfying the hypothesis of Theorem 5 and for any non-negative integer $n \geq u_0 - \frac{a}{b} r - \frac{1}{2}$:

$$\mathrm{lcm}\{u_0, \dots, u_n\} \ \geq \ \frac{1}{\pi} \sqrt{r} (r+1)^{n-1+\frac{u_0}{r}}.$$

We introduce a non-negative parameter δ and the arithmetic progression $(u_k)_k$ (depending on δ) with first term $u_0 := ab\delta + 1$ and difference $r := b^2\delta$. The integers u_0 and r are coprime because they verify the Bézout identity $(1 - ab\delta)u_0 + a^2\delta r = 1$. The sequence $(u_k)_k$ thus satisfies all the hypotheses of Theorem 5. Since the integer $n = 1$ satisfies $n \geq u_0 - \frac{a}{b} r - \frac{1}{2} = \frac{1}{2}$, we must have:

$$\mathrm{lcm}\{u_0, u_1\} \ \geq \ \frac{1}{\pi} \sqrt{r} (r+1)^{\frac{u_0}{r}}. \tag{10}$$

Further, we have

$$\mathrm{lcm}\{u_0, u_1\} = u_0 u_1 = (ab\delta + 1)\left((ab + b^2)\delta + 1\right) = O(\delta^2)$$

and

$$\frac{1}{\pi} \sqrt{r} (r+1)^{\frac{u_0}{r}} = \frac{1}{\pi} b \sqrt{\delta} (b^2\delta + 1)^{\frac{a}{b} + \frac{1}{b^2\delta}} = O\left(\delta^{\frac{a}{b} + \frac{1}{2}}\right).$$

But since $\frac{a}{b} + \frac{1}{2} > 2$, The relation (10) cannot holds for δ sufficiently large. Contradiction. Theorem 7 follows. ■

Proof of Theorem 8. Let us prove the assertion 1) of Theorem 8. The fact that the constant C of this assertion is greater than or equal to $\frac{1}{\pi}$ is an immediate consequence of the part 4) of Theorem 5. In order to prove the upper bound $C \leq \frac{3}{2}$, we introduce a parameter $\delta \in \mathbb{N}$ and the arithmetic sequence $(u_k)_k$ (depending on δ), with first term $u_0 := 3\delta + 2$ and difference $r := 2\delta + 1$. The integers u_0 and r are coprime because they verify the Bézout identity $2u_0 - 3r = 1$. So, this sequence $(u_k)_k$ satisfies all the hypotheses of Theorem 5. Since $u_0 - \frac{3r+1}{2} = 0$, we must have for any non-negative integer n: $\mathrm{lcm}\{u_0, \dots, u_n\} \geq C\sqrt{r}(r+1)^{n-1+\frac{u_0}{r}}$, in particular (for $n = 0$): $u_0 \geq C\sqrt{r}(r+1)^{\frac{u_0}{r}-1}$, hence:

$$C \leq \frac{u_0}{\sqrt{r}(r+1)^{\frac{u_0}{r}-1}}.$$

Since this last upper bound holds for any $\delta \in \mathbb{N}$, we finally deduce that:

$$C \leq \lim_{\delta \to +\infty} \frac{u_0}{\sqrt{r}(r+1)^{\frac{u_0}{r}-1}} = \lim_{\delta \to +\infty} \frac{3\delta + 2}{\sqrt{2\delta+1}(2\delta+2)^{\frac{\delta+1}{2\delta+1}}} = \frac{3}{2}$$

as required.

Now, let us prove the assertion 2) of Theorem 8. Let n_0 be a fixed non-negative integer. As above, the lower bound $C(n_0) \geq \frac{1}{\pi}$ is an immediate consequence of the part 4) of Theorem 5. In order to prove the upper bound of Theorem 8 for the constant $C(n_0)$, we choose an integer n_1 such that $n_0 + 3 \leq n_1 \leq 2n_0 + 6$ and that $(n_1 + 1)$ is prime (this is possible from the Bertrand postulate). Then, we introduce a parameter $\delta \in \mathbb{N}$ which is not a multiple of $(n_1 + 1)$ and the arithmetic progression $(u_k)_k$ (depending on δ), with first term $u_0 := 3\delta n_1!$ and difference $r := 2\delta n_1! + n_1 + 1$. These integers u_0 and r are coprime. Indeed, a common divisor $d \geq 1$ between u_0 and r divides $3r - 2u_0 = 3(n_1 + 1)$, thus it divides $\gcd\{u_0, 3(n_1 + 1)\} = 3\gcd\{\delta n_1!, n_1 + 1\}$. Further, the fact that $(n_1 + 1)$ is prime implies that $(n_1 + 1)$ is coprime with $n_1!$, moreover since δ is not a multiple of $(n_1 + 1)$, the integer $(n_1 + 1)$ also is coprime with δ. It follows that $(n_1 + 1)$ is coprime with the product $\delta n_1!$. Hence d divides 3. But since 3 divides $2\delta n_1!$ (because $n_1 \geq 3$) and 3 doesn't divide $n_1 + 1$ (because $n_1 + 1$ is a prime number ≥ 5) then 3 cannot divide the sum $2\delta n_1! + (n_1 + 1) = r$, which proves that $d \neq 3$. Consequently $d = 1$, that is u_0 and r are coprime effectively. The sequence $(u_k)_k$ which we have introduced thus satisfies all the hypotheses of Theorem 5. Since $n_1 \geq \max\{n_0, u_0 - \frac{3r+1}{2}\}$ (because $n_1 \geq n_0 + 3$ and $u_0 - \frac{3r+1}{2} = -\frac{3}{2}n_1 - 2 < 0$), then we must have $\mathrm{lcm}\{u_0, \dots, u_{n_1}\} \geq C(n_0)\sqrt{r}(r+1)^{n_1-1+\frac{u_0}{r}}$. This gives:

$$C(n_0) \leq \frac{\mathrm{lcm}\{u_0, \dots, u_{n_1}\}}{\sqrt{r}(r+1)^{n_1-1+\frac{u_0}{r}}}.$$

Now, since u_0 is a multiple of $n_1!$, we have from Theorem 4: $\mathrm{lcm}\{u_0, \dots, u_{n_1}\} = \frac{u_0 \dots u_{n_1}}{n_1!}$. Hence:

$$C(n_0) \leq \frac{u_0 \dots u_{n_1}}{n_1!\sqrt{r}(r+1)^{n_1-1+\frac{u_0}{r}}}.$$

Since this last upper bound of $C(n_0)$ holds for any $\delta \in \mathbb{N}$ which is not a multiple of $(n_1 + 1)$, then we deduce that:

$$C(n_0) \;\leq\; \lim_{\substack{\delta \to +\infty \\ \delta \not\equiv 0 \bmod (n_1+1)}} \frac{u_0 \dots u_{n_1}}{n_1! \sqrt{r} (r+1)^{n_1 - 1 + \frac{u_0}{r}}}. \tag{11}$$

Let us calculate the limit from the right-hand side of (11). We have:

$$\begin{aligned} u_0 \dots u_{n_1} = \prod_{k=0}^{n_1} (u_0 + kr) &= \prod_{k=0}^{n_1} \left\{ (2k+3) n_1! \delta + k(n_1 + 1) \right\} \\ &\sim_{+\infty} \left(n_1!^{n_1+1} \prod_{k=0}^{n_1} (2k+3) \right) \delta^{n_1+1} \end{aligned}$$

and:

$$\begin{aligned} \sqrt{r}(r+1)^{n_1 - 1 + \frac{u_0}{r}} &= (2\delta n_1! + n_1 + 1)^{1/2} \, (2\delta n_1! + n_1 + 2)^{n_1 - 1 + \frac{3\delta n_1!}{2\delta n_1! + n_1 + 1}} \\ &\sim_{+\infty} (2\delta n_1!)^{n_1+1}. \end{aligned}$$

Then:

$$\frac{u_0 \dots u_{n_1}}{n_1! \sqrt{r} (r+1)^{n_1 - 1 + \frac{u_0}{r}}} \;\sim_{+\infty}\; \frac{\prod_{k=0}^{n_1} (2k+3)}{2^{n_1+1} n_1!} = \frac{(n_1+1)(n_1 + \frac{3}{2})}{4^{n_1}} \binom{2n_1 + 1}{n_1}.$$

In the other words:

$$\lim_{\delta \to +\infty} \frac{u_0 \dots u_{n_1}}{n_1! \sqrt{r} (r+1)^{n_1 - 1 + \frac{u_0}{r}}} \;=\; \frac{(n_1+1)(n_1 + \frac{3}{2})}{4^{n_1}} \binom{2n_1 + 1}{n_1}.$$

It is easy to show (by induction on k) that for any non-negative integer k, we have $\binom{2k+1}{k} < \sqrt{2} \frac{4^k}{\sqrt{k + \frac{3}{2}}}$. Using this estimate for $k = n_1$, we finally deduce that:

$$\begin{aligned} \lim_{\delta \to +\infty} \frac{u_0 \dots u_{n_1}}{n_1! \sqrt{r} (r+1)^{n_1 - 1 + \frac{u_0}{r}}} &< \sqrt{2}(n_1 + 1) \sqrt{n_1 + \frac{3}{2}} \\ &< 4(n_0 + 4)\sqrt{n_0 + 4} \quad \text{(because } n_1 \leq 2n_0 + 6\text{)}. \end{aligned}$$

The upper bound $C(n_0) < 4(n_0 + 4)\sqrt{n_0 + 4}$ follows by substituting this last estimate into (11). This completes the proof of Theorem 8. ■

Proof of Theorem 9. We first prove Theorem 9 in the particular case $m = 0$. We deduce the general case of the same Theorem by shifting the terms of the sequence $\mathbf{u} = (u_k)_k$. Let $\mathbf{u}$ be a sequence as in Theorem 9.

• The case $m = 0$: From Theorem 2, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number

$$R := \frac{u_0 \dots u_n}{\mathrm{lcm}\left\{\prod_{0 \leq i \leq n, i \neq j} (u_i - u_j) \ ; \ j = 0, \dots, n\right\}}. \tag{12}$$

Now, since we have for any $i, j \in \mathbb{N}$:

$$u_i - u_j = \{ai(i+t) + b\} - \{aj(j+t) + b\} = a(i-j)(i+j+t),$$

then:

$$\begin{aligned} \prod_{0 \leq i \leq n, i \neq j} (u_i - u_j) &= \prod_{0 \leq i \leq n, i \neq j} \{a(i-j)(i+j+t)\} \\ &= a^n \prod_{0 \leq i \leq n, i \neq j} (i-j). \prod_{0 \leq i \leq n, i \neq j} (i+j+t) \\ &= \begin{cases} a^n (-1)^j \frac{(n-j)!(n+j)!}{2} & \text{if } t = 0 \\ a^n (-1)^j \frac{(n-j)!(n+j+t)!}{\varphi(j,t)} \frac{1}{2j+t} & \text{if } t \geq 1 \end{cases}, \end{aligned}$$

where $\varphi(j,t) := 1$ if $t = 1$ and $\varphi(j,t) := (j+1)\dots(j+t-1)$ if $t \geq 2$. Since $(n-j)!(n+j+t)!$ divides $(2n+t)!$ (because $\frac{(2n+t)!}{(n-j)!(n+j+t)!} = \binom{2n+t}{n-j} \in \mathbb{N}$) and (if $t \geq 1$) the integer $\varphi(j,t)$ is a multiple of $(t-1)!$ (because $\frac{\varphi(j,t)}{(t-1)!} = \binom{j+t-1}{t-1} \in \mathbb{N}$), then the product $\prod_{0 \leq i \leq n, i \neq j} (u_i - u_j)$ divides the integer (which does not depend on j):

$$f(t,n) := \begin{cases} a^n \frac{(2n)!}{2} & \text{if } t = 0 \\ a^n \frac{(2n+t)!}{(t-1)!} & \text{if } t \geq 1 \end{cases}.$$

Since j is arbitrary in $\{0, \dots, n\}$, then the integer $\mathrm{lcm}\{\prod_{0 \leq i \leq n, i \neq j} (u_i - u_j); j = 0, \dots, n\}$ divides the integer $f(t,n)$. It follows that the rational number R (of (12)) is a multiple of the rational number $\frac{u_0 \dots u_n}{f(t,n)} = \frac{A_{\mathbf{u}}(t,0,n)}{a^n}$. Consequently, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number $\frac{A_{\mathbf{u}}(t,0,n)}{a^n}$. Finally, since each term of the sequence $\mathbf{u}$ is coprime with a (because $\gcd\{a, b\} = 1$), we conclude from the Gauss Lemma that the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number $A_{\mathbf{u}}(t, 0, n)$ as required.

• The general case ($m \in \mathbb{N}$): Let us consider the new sequence $\mathbf{v} = (v_k)_{k \in \mathbb{N}}$ with general term:

$$v_k := u_{k+m} = a'k(k+t') + b',$$

where $a' := a$, $t' := 2m + t$ and $b' := am(m + t) + b$.
Since these integers a', t' and b' verify $a' \geq 1$, $t' \geq 0$ and $\gcd\{a', b'\} = \gcd\{a, b\} = 1$ obviously, then the sequence $\mathbf{v}$ satisfies all the hypotheses of Theorem 9. Thus, from the particular case (proved above) of this Theorem, the integer $\mathrm{lcm}\{v_0, \dots, v_{n-m}\} = \mathrm{lcm}\{u_m, \dots, u_n\}$ is a multiple of the rational number $A_{\mathbf{v}}(t', 0, n - m) = A_{\mathbf{u}}(t, m, n)$ which provides the desired conclusion. $\blacksquare$

Proof of Corollary 10. From Theorem 9, the integer $\mathrm{lcm}\{u_0, \dots, u_n\}$ is a multiple of the rational number:

$$A_{\mathbf{u}}(t, 0, n) \;:=\; \begin{cases} 2\frac{u_0 \dots u_n}{(2n)!} & \text{if } t = 0 \\ (t-1)!\frac{u_0 \dots u_n}{(2n+t)!} & \text{if } t \geq 1 \end{cases} .$$

Let us get a lower bound for this last number which doesn't depend on the terms of the sequence $\mathbf{u}$. Using the obvious lower bounds $u_k \geq ak(k+t)$ $(1 \leq k \leq n)$, we have:

$$u_0 \dots u_n \;\geq\; b\{a.1.(1+t)\}\{a.2.(2+t)\} \dots \{a.n.(n+t)\} = ba^n \frac{n!(n+t)!}{t!},$$

then:

$$A_{\mathbf{u}}(t, 0, n) \;\geq\; \begin{cases} 2b\frac{a^n}{\binom{2n}{n}} & \text{if } t = 0 \\ \frac{b}{t}\frac{a^n}{\binom{2n+t}{n}} & \text{if } t \geq 1 \end{cases} \;\geq\; \begin{cases} 2b\left(\frac{a}{4}\right)^n & \text{if } t = 0 \\ \frac{b}{t2^t}\left(\frac{a}{4}\right)^n & \text{if } t \geq 1 \end{cases}$$

(because $\binom{2n}{n} \leq 2^{2n} = 4^n$ and $\binom{2n+t}{n} \leq 2^{2n+t} = 2^t 4^n$). The lower bound of Corollary 10 follows. $\blacksquare$

Proof of Theorem 11. Theorem 11 is only a combination of the results of Theorems 3 and 4 which we apply for the arithmetic progression $(u_\ell)_{\ell \in \mathbb{N}}$ with general term $u_\ell = \ell + n$ (where $n \in \mathbb{N}$ is fixed). $\blacksquare$

Proof of Theorem 12. Let us prove the first assertion of Theorem 12. Giving k a non-negative integer and n a positive integer, we easily show that for any non-negative integer $j \leq k$, we have:

$$n\binom{n+k}{k}\binom{k}{j} \;=\; (n+j)\binom{n+j-1}{j}\binom{n+k}{k-j}.$$

It follows that the integer $\mathrm{lcm}\Big\{n\binom{n+k}{k}\binom{k}{j}\ ;\ j = 0, \dots, k\Big\} = n\binom{n+k}{k}\mathrm{lcm}\{\binom{k}{0}, \dots, \binom{k}{k}\}$ is a multiple of each integer $n + j$ $(0 \leq j \leq k)$. Then it is a multiple of $\mathrm{lcm}\{n, n+1, \dots, n+k\}$, as required.

Now, in order to prove the second assertion of Theorem 12, we introduce the sequence of maps $(g_k)_{k\in\mathbb{N}}$ of $\mathbb{N}^*$ into $\mathbb{N}^*$ which is defined by:

$$g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}\{n, n+1, \dots, n+k\}} \qquad (\forall k \in \mathbb{N}, \forall n \in \mathbb{N}^*).$$

Let us show that $(g_k)_k$ satisfies the induction relation:

$$g_k(n) = \gcd\{k!, (n+k)g_{k-1}(n)\} \qquad (\forall (k,n) \in \mathbb{N}^{*2}). \tag{13}$$

For any pair of positive integers (k, n), we have:

$$\begin{aligned}
g_k(n) \ &:= \ \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}\{n, n+1, \dots, n+k\}} \\
&= \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}\left\{\mathrm{lcm}\{n, n+1, \dots, n+k-1\}, n+k\right\}} \\
&= \frac{n(n+1)\dots(n+k)}{\frac{\mathrm{lcm}\{n,n+1,\dots,n+k-1\}.(n+k)}{\gcd\{\mathrm{lcm}\{n,n+1,\dots,n+k-1\},n+k\}}} \\
&= \frac{n(n+1)\dots(n+k-1)}{\mathrm{lcm}\{n, n+1, \dots, n+k-1\}} \gcd\left\{\mathrm{lcm}\{n, n+1, \dots, n+k-1\}, n+k\right\} \\
&= \gcd\left\{n(n+1)\dots(n+k-1), (n+k)g_{k-1}(n)\right\}.
\end{aligned}$$

Then, the relation (13) follows by remarking that the product $n(n+1)\dots(n+k-1)$ is a multiple of $k!$ (because $\frac{n(n+1)\dots(n+k-1)}{k!} = \binom{n+k-1}{k} \in \mathbb{N}$) and that $g_k(n)$ divides $k!$ (according to Theorem 11).

Now, giving a non-negative integer k, by reiterating the relation (13) several times, we obtain:

$$\begin{aligned}
g_k(n) &= \gcd\{k!, (n+k)g_{k-1}(n)\} \\
&= \gcd\{k!, (n+k)(k-1)!, (n+k)(n+k-1)g_{k-2}(n)\} \\
&\ \ \vdots \\
&= \gcd\{k!, (n+k)\cdot(k-1)!, (n+k)(n+k-1)\cdot(k-2)!, \dots, \\
&\qquad\qquad (n+k)(n+k-1)\cdots(n+k-\ell)g_{k-\ell-1}(n)\}
\end{aligned}$$

for any positive integer n and any non-negative integer $\ell \leq k-1$. In particular, for $\ell = k-1$, since $g_0 \equiv 1$, we have for any positive integer n:

$$\begin{aligned}
g_k(n) = \gcd\{&k!, (n+k)\cdot(k-1)!, (n+k)(n+k-1)\cdot(k-2)!, \dots, \\
&(n+k)(n+k-1)\cdots(n+1).0!\}
\end{aligned} \tag{14}$$

Now, if n is a given positive integer satisfying the congruence $n+k+1 \equiv 0 \bmod(k!)$, we have:

$$n+k \equiv -1 \bmod(k!)\ ,\ (n+k)(n+k-1) \equiv (-1)^2 2! \bmod(k!)\ ,\ \dots\ ,$$
$$(n+k)(n+k-1)\cdots(n+1) \equiv (-1)^k k! \bmod(k!);$$

consequently, the relation (14) gives:

$$g_k(n) = \gcd\{k!, 1!(k-1)!, 2!(k-2)!, \dots, k!0!\}.$$

Hence:

$$\begin{aligned}\frac{k!}{g_k(n)} &= \frac{k!}{\gcd\{0!k!, 1!(k-1)!, \dots, k!0!\}}\\ &= \operatorname{lcm}\left\{\frac{k!}{0!k!}, \frac{k!}{1!(k-1)!}, \dots, \frac{k!}{k!0!}\right\}\\ &= \operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}.\end{aligned}$$

But on the other hand, according to the definition of $g_k(n)$, we have:

$$\frac{k!}{g_k(n)} = \frac{\operatorname{lcm}\{n, n+1, \dots, n+k\}}{n\binom{n+k}{k}}.$$

We thus conclude that:

$$\operatorname{lcm}\{n, n+1, \dots, n+k\} = n\binom{n+k}{k}\operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}$$

which gives the second assertion of Theorem 12 and completes this proof. ■

Open Problem. By using the relation (13), we can easily show (by induction on k) that for any non-negative integer k, the map g_k which we have introduced above is periodic of period $k!$. In other words, the map g_k ($k \in \mathbb{N}$) is defined modulo $k!$. Then, for k fixed in $\mathbb{N}$, it is sufficient to calculate $g_k(n)$ for the $k!$ first values of n ($n = 1, \dots, k!$) to have all the values of g_k. Consequently, the relation (13) is a practical mean which permits to determinate step by step all the values of the maps g_k. By proceeding in this way, we obtain: $g_0(n) \equiv g_1(n) \equiv 1$ (obviously),

$$g_2(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}\ ,\quad g_3(n) = \begin{cases} 6 & \text{if } n \equiv 0 \bmod(3) \\ 2 & \text{otherwise} \end{cases}\ ,\quad \dots\text{etc.}$$

This calculation point out that the smallest period of the map g_3 is equal to $3 (\neq 3!)$. This lead us to ask the following interesting open question:

> Giving k a non-negative integer, what is the smallest period for the map g_k?

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