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A new generalization of the Genocchi numbers and its consequence on the Bernoulli polynomials

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Abstract

This paper presents a new generalization of the Genocchi numbers and the Genocchi theorem. As consequences, we obtain some important families of integer-valued polynomials those are closely related to the Bernoulli polynomials. Denoting by $(B_n)_{n\in\mathbb{N}}$ the sequence of the Bernoulli numbers and by $(B_n(X))_{n\in\mathbb{N}}$ the sequence of the Bernoulli polynomials, we especially obtain that for any natural number n, the reciprocal polynomial of the polynomial $\left(B_n(X) - B_n\right)$ is integer-valued.

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1 Introduction and Notations

Throughout this paper, we let $\mathbb{N}^*$ denote the set of positive integers. For a given prime number p, we let ϑ_p denote the usual p-adic valuation. The rational numbers x satisfying $\vartheta_p(x) \geq 0$ are called *p-integers*; they constitute a subring of $\mathbb{Q}$, usually denoted by $\mathbb{Z}_{(p)}$. For a given rational number r, we let $\mathrm{den}(r)$ denote the denominator of r; that is the smallest positive integer d such that $dr \in \mathbb{Z}$.

Next, we let $\mathbb{Q}[X]$ denote the ring of polynomials in X with coefficients in $\mathbb{Q}$. If $P \in \mathbb{Q}[X]$, we let $\deg P$ denote the degree of P. We call the *reciprocal polynomial* of a polynomial $P \in \mathbb{Q}[X]$ the polynomial P^* ($\in \mathbb{Q}[X]$) obtained by reversing the order of the coefficients of P; for example $(2X^3 + 5X^2 + 7X + 3)^* = 3X^3 + 7X^2 + 5X + 2$. It is easy to show that for any $P \in \mathbb{Q}[X]$, we have $P^*(X) = X^{\deg P} P(\frac{1}{X})$. We let Δ denote the forward difference operator on $\mathbb{Q}[X]$; that is $(\Delta P)(X) := P(X+1) - P(X)$ ($\forall P \in \mathbb{Q}[X]$). A polynomial $P \in \mathbb{Q}[X]$ whose value $P(n)$ is an integer for every integer n (i.e., $P(\mathbb{Z}) \subset \mathbb{Z}$) is called an *integer-valued polynomial*. The set of integer-valued polynomials is denoted by $\mathrm{Int}(\mathbb{Z})$ and forms a $\mathbb{Z}$-algebra (under the usual operations on polynomials). It is known (see, e.g., [2, 8]) that $\mathrm{Int}(\mathbb{Z})$ (seen as a $\mathbb{Z}$-module) is free with infinite rank and has as a basis the sequence of polynomials $\binom{X}{n}$ ($n \in \mathbb{N}$), where $\binom{X}{n} := \frac{X(X-1)\cdots(X-n+1)}{n!}$ ($\forall n \in \mathbb{N}$). An exhaustive study of the integer-valued polynomials (including the integer-valued polynomials on a general domain) is given in the book of Cahen and Chabert [2].

Further, *the Bernoulli polynomials* $B_n(X)$ ($n \in \mathbb{N}$) can be defined by the exponential generating function:

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(X) \frac{t^n}{n!}$$

and *the Bernoulli numbers* B_n are the values of the Bernoulli polynomials at $X = 0$; that is $B_n := B_n(0)$ ($\forall n \in \mathbb{N}$). To mark the difference between the Bernoulli polynomials and the Bernoulli numbers, we always put the indeterminate X in evidence when it comes to polynomials. The Bernoulli polynomials and numbers have many important and remarkable properties; an elementary presentation (but quite rich) can be found in the book of Nielsen [7]. It is known for example that $\deg B_n(X) = n$ ($\forall n \in \mathbb{N}$) and that $B_n = 0$ for any odd integer $n \geq 3$.

Throughout this paper, we deal with *formal power series* with rational coefficients. We denote by $\mathbb{Q}[[t]]$ the ring of formal power series in t with coefficients in $\mathbb{Q}$. An element S of $\mathbb{Q}[[t]]$ is always represented as

$$S(t) := \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!},$$

where $a_n \in \mathbb{Q}$ ($\forall n \in \mathbb{N}$). The a_n's are called *the differential coefficients* of S (because it is immediate that each a_n is the n^{th} derivative of S at 0). If the a_n's are all integers, we say that S is an *IDC-series* (IDC abbreviates the expression "with Integral Differential Coefficients"). Many usual functions are IDC-series; we can cite for example the functions $x \mapsto e^x$, $x \mapsto \sin x$, $x \mapsto \cos x$, $x \mapsto \ln(1+x)$, and so on. The sum of two IDC-series is obviously an IDC-series. The product of two IDC-series is also an IDC-series. Indeed, if S_1 and S_2 are two IDC-series with

$$S_1(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!} \quad , \quad S_2(t) = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}$$

(so $a_n, b_n \in \mathbb{Z}$, $\forall n \in \mathbb{N}$) then we have

$$\begin{aligned} S_1(t)S_2(t) &= \left(\sum_{k=0}^{+\infty} a_k \frac{t^k}{k!}\right)\left(\sum_{\ell=0}^{+\infty} b_\ell \frac{t^\ell}{\ell!}\right) = \sum_{k,\ell \in \mathbb{N}} \frac{(k+\ell)!}{k!\ell!} a_k b_\ell \frac{t^{k+\ell}}{(k+\ell)!} \\ &= \sum_{n=0}^{+\infty} \left(\sum_{\substack{k,\ell \in \mathbb{N} \\ k+\ell=n}} \frac{n!}{k!\ell!} a_k b_\ell\right) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} c_n \frac{t^n}{n!}, \end{aligned}$$

where

$$c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \qquad (\forall n \in \mathbb{N}).$$

Since $a_k, b_k \in \mathbb{Z}$ ($\forall n \in \mathbb{N}$) then $c_n \in \mathbb{Z}$ ($\forall n \in \mathbb{N}$), showing that $S_1 S_2$ is an IDC-series.

Showing that a given function is an IDC-series is not always easy. The more famous example is perhaps the function $t \mapsto \frac{2t}{e^t+1}$ whose expansion into a power series is

$$\frac{2t}{e^t+1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!},$$

where the G_n's are called *the Genocchi numbers* and have been studied by several authors (see, e.g., [3, 4, 5, 9]). An important theorem of Genocchi [6] states that the G_n's are all integers; equivalently, the function $t \mapsto \frac{2t}{e^t+1}$ is an IDC-series. A familiar proof of this curious result uses the expression of the G_n's in terms of the Bernoulli numbers:

$$G_n = -2(2^n - 1)B_n \qquad (\forall n \in \mathbb{N}),$$

together with the von Staudt-Clausen theorem and the Fermat little theorem.

In this paper, we generalize the Genocchi numbers by considering for a given integer $a \geq 2$, the function $t \mapsto \frac{at}{e^{(a-1)t}+e^{(a-2)t}+\cdots+e^t+1}$ and its expansion into a power series:

$$\frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}.$$

For $a = 2$, we simply obtain the usual Genocchi numbers; that is $G_{n,2} = G_n$ ($\forall n \in \mathbb{N}$). In our main Theorem 3.1, we prove that the $G_{n,a}$'s are all integers, which generalizes the Genocchi theorem. Next, by interpolating the numbers $G_{n,a}$ ($a \geq 2$), for a fixed $n \in \mathbb{N}$, we derive some important families of integer-valued polynomials which are closely related to the Bernoulli polynomials. We particularly obtain that for any natural number n, the polynomial $(B_n(X) - B_n)^*$ is integer-valued.

2 Some preliminaries on the IDC-series

In this section, we present some selected elementary properties of the IDC-series. We begin with the following proposition:

Proposition 2.1. *Let f be an IDC-series. Then $\frac{1}{f}$ is an IDC-series if and only if $f(0) = \pm 1$.*

Proof. Write

$$f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!},$$

where $a_n \in \mathbb{Z}$, $\forall n \in \mathbb{N}$.

• If $\frac{1}{f}$ is an IDC-series then we have $(\frac{1}{f})(0) = \frac{1}{f(0)} \in \mathbb{Z}$, which is possible if and only if $f(0) = \pm 1$

(since $f(0) = a_0 \in \mathbb{Z}$).

• Conversely, suppose that $f(0) = \pm 1$ (that is $a_0 = \pm 1$) and let us show that $\frac{1}{f}$ is an IDC-series. The fact that $f \in \mathbb{Q}[[t]]$ and $f(0) \neq 0$ implies that $\frac{1}{f} \in \mathbb{Q}[[t]]$; so let

$$\frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!},$$

where $b_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$. Thus, we have

$$\left(\sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}\right) = 1.$$

Then, by identifying the differential coefficients in both power series of the last identity, we obtain that:

$$\begin{cases} a_0 b_0 = 1 \\ \displaystyle\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = 0 \quad (\forall n \geq 1) \end{cases}.$$

Hence

$$\begin{cases} b_0 = \frac{1}{a_0} \\ b_n = -\frac{1}{a_0}\left[\binom{n}{1} b_{n-1} a_1 + \binom{n}{2} b_{n-2} a_2 + \cdots + \binom{n}{n} b_0 a_n\right] \quad (\forall n \geq 1) \end{cases},$$

showing that $b_0 \in \mathbb{Z}$ (since $a_0 = \pm 1$ by hypothesis) and then (by a simple induction on n) that $b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Thus $\frac{1}{f}$ is an IDC-series, as required. Our proof is complete. □

From Proposition 2.1, we derive the following corollary:

Corollary 2.2. *Let $f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$ be an IDC-series with $a_0 \neq 0$. Then the formal power series $\frac{a_0}{f(a_0 t)}$ is also an IDC-series.*

Proof. We have

$$\frac{f(a_0 t)}{a_0} = \frac{1}{a_0} \sum_{n=0}^{+\infty} a_n \frac{(a_0 t)^n}{n!} = 1 + \sum_{n=1}^{+\infty} a_n a_0^{n-1} \frac{t^n}{n!},$$

showing that $\frac{f(a_0 t)}{a_0}$ is an IDC-series with the first coefficient equal to 1. According to Proposition 2.1, it follows that $\frac{1}{f(a_0 t)/a_0} = \frac{a_0}{f(a_0 t)}$ is also an IDC-series. This achieves the proof. □

Finally, from Corollary 2.2, we derive the following corollary which is essential for our purpose.

Corollary 2.3. *Let $f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$ be an IDC-series with $a_0 \neq 0$ and let $\frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!} \in \mathbb{Q}[[t]]$ be the reciprocal of the formal power series f. Then, for all $n \in \mathbb{N}$, the denominator of the rational number b_n divides the integer a_0^{n+1}.*

Proof. From $\frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}$, we derive that:

$$\frac{a_0}{f(a_0 t)} = a_0 \sum_{n=0}^{+\infty} b_n \frac{(a_0 t)^n}{n!} = \sum_{n=0}^{+\infty} b_n a_0^{n+1} \frac{t^n}{n!}.$$

But, according to Corollary 2.2, we know that $\frac{a_0}{f(a_0 t)}$ is an IDC-series; equivalently, we have that $b_n a_0^{n+1} \in \mathbb{Z}$ ($\forall n \in \mathbb{N}$). Consequently, the denominator of each of the rational numbers b_n ($n \in \mathbb{N}$) is a divisor of a_0^{n+1}, as required. The corollary is proved. □

3 The main result

Our main result is the following:

Theorem 3.1. *Let $a \geq 2$ be an integer. Then for any positive integer n, the number $G_{n,a}$ is an integer.*

If we take $a = 2$ in Theorem 3.1, we obtain the Genocchi original theorem.

The method of proving Theorem 3.1 consists to show that for any prime number p, we have $\vartheta_p(G_{n,a}) \geq 0$ ($a \geq 2$, $n \in \mathbb{N}^*$). To do so, we distinguish two cases according as p does or does not divide a. We begin with the following proposition:

Proposition 3.2. *Let a and n be two positive integers with $a \geq 2$. Then the denominator of the rational number $G_{n,a}$ divides a^{n-1}.*

Proof. By applying Corollary 2.3 for the IDC-series

$$f(t) \; := \; e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1 \; = \; a + \sum_{n=1}^{+\infty} \left(1^n + 2^n + \cdots + (a-1)^n\right) \frac{t^n}{n!},$$

we obtain that the expansion of $\frac{1}{f}$ into a power series has the form

$$\frac{1}{f(t)} \; = \; \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}, \tag{3.1}$$

where, for all $n \in \mathbb{N}$, we have $b_n \in \mathbb{Q}$ and $\mathrm{den}(b_n) \mid a^{n+1}$. Then, by multiplying the two sides of (3.1) by at, we get

$$\frac{at}{f(t)} \; = \; \sum_{n=0}^{+\infty} a b_n \frac{t^{n+1}}{n!} \; = \; \sum_{n=1}^{+\infty} a b_{n-1} \frac{t^n}{(n-1)!} \; = \; \sum_{n=1}^{+\infty} a n b_{n-1} \frac{t^n}{n!}.$$

But since we have on the other hand $\frac{at}{f(t)} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}$, we deduce that

$$G_{n,a} \; = \; a n b_{n-1} \qquad\qquad (\forall n \in \mathbb{N}^*).$$

Now, for a given $n \in \mathbb{N}^*$, we have that $\mathrm{den}(b_{n-1}) \mid a^n$; thus $\mathrm{den}(a n b_{n-1}) \mid a^{n-1}$; that is $\mathrm{den}(G_{n,a}) \mid a^{n-1}$. This completes the proof. □

From Proposition 3.2, we immediately derive the following corollary:

Corollary 3.3. *Let a and n be two positive integers with $a \geq 2$. Then for any prime number p not dividing a, we have*

$$\vartheta_p\left(G_{n,a}\right) \; \geq \; 0.$$ □

Now, we are going to establish the analog of Corollary 3.3 for the prime numbers p dividing the considered number a. For this purpose, we first need the following proposition:

Proposition 3.4. *Let $a \geq 2$ be an integer. Then for all positive integer n, we have*

$$G_{n,a} + \sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} \; = \; 1.$$

Proof. From the definition of the numbers $G_{n,a}$, we have

$$\left(\sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}\right) \left(\sum_{n=0}^{+\infty} \frac{a^n}{n+1} \frac{t^n}{n!}\right) = \frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} \cdot \frac{e^{at} - 1}{at} \; = \; e^t - 1 \; = \; \sum_{n=1}^{+\infty} \frac{t^n}{n!};$$

that is

$$\left(\sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}\right) \left(\sum_{n=0}^{+\infty} \frac{a^n}{n+1} \frac{t^n}{n!}\right) \; = \; \sum_{n=1}^{+\infty} \frac{t^n}{n!}. \tag{3.2}$$

So, for a given $n \in \mathbb{N}^*$, the identification of the n^{th} differential coefficients in the two hand-sides of (3.2) gives

$$\sum_{k=0}^{n} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} \; = \; 1,$$

which is nothing else the required identity (since $G_{0,a} = 0$). □

Next, we have the following lemma:

Lemma 3.5. *Let $a \geq 2$ be an integer. Then for all prime number p dividing a and all natural number k, we have*

$$\vartheta_p\left(\frac{a^k}{k+1}\right) \; \geq \; 0.$$

Proof. Let p be a prime number dividing a (so $\vartheta_p(a) \geq 1$) and k be a natural number. Since $k + 1 \leq 2^k \leq p^k$ then we have $\vartheta_p(k+1) \leq k$. Hence

$$\vartheta_p\left(\frac{a^k}{k+1}\right) \; = \; k\vartheta_p(a) - \vartheta_p(k+1) \; \geq \; k - \vartheta_p(k+1) \; \geq \; 0,$$

as required. □

From Proposition 3.4 and Lemma 3.5, we derive the following corollary:

Corollary 3.6. *Let a and n be two positive integers with $a \geq 2$. Then for any prime number p dividing a, we have*

$$\vartheta_p\left(G_{n,a}\right) \; \geq \; 0.$$

Proof. Let p be a prime number dividing a. To prove that $\vartheta_p(G_{n,a}) \geq 0$, we argue by induction on $n \in \mathbb{N}^*$ and use the identity of Proposition 3.4 together with Lemma 3.5.

- For $n = 1$, we have $G_{1,a} = 1$, so $\vartheta_p(G_{1,a}) = 0 \geq 0$.

• Let $n \geq 2$ be an integer. Suppose that $\vartheta_p(G_{m,a}) \geq 0$ for any positive integer $m < n$ and show that $\vartheta_p(G_{n,a}) \geq 0$. By Proposition 3.4, we have

$$G_{n,a} = -\sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} + 1.$$

Since the binomial coefficients are known to be integers, the numbers $G_{n-k,a}$ $(1 \leq k \leq n-1)$ are p-integers (by the induction hypothesis) and the numbers $\frac{a^k}{k+1}$ $(1 \leq k \leq n-1)$ are p-integers (by Lemma 3.5) then the sum $-\sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} + 1$ is a p-integer; that is $\vartheta_p(G_{n,a}) \geq 0$, as required. This achieves the induction, and hence, the proof. □

The proof of our main result is now immediate:

Proof of Theorem 3.1. Let a and n be two positive integers with $a \geq 2$. According to Corollaries 3.3 and 3.6, we have for any prime number p: $\vartheta_p(G_{n,a}) \geq 0$. Thus the number $G_{n,a}$ is an integer. Our main result is proved. □

4 Some consequences of the main result

For the following, we extend the numbers $G_{n,a}$ to non-integer values of a. Precisely, we define $\mathcal{G}_n(x)$ $(n \in \mathbb{N}, x \in \mathbb{R})$ as the coefficients occurring on the right-hand side of the identity:

$$\frac{xt}{e^{xt}-1}(e^t - 1) = \sum_{n=0}^{+\infty} \mathcal{G}_n(x) \frac{t^n}{n!},$$

where it is understood that $\mathcal{G}_n(0) = \lim_{x \to 0} \mathcal{G}_n(x) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{otherwise} \end{cases}$ (because $\lim_{x \to 0} \frac{xt}{e^{xt}-1}(e^t - 1) = e^t - 1$). Then it is immediate that $\mathcal{G}_n(a) = G_{n,a}$ if $n \in \mathbb{N}$ and a is an integer ≥ 2. The following proposition shows that for any $n \in \mathbb{N}$, the function $x \mapsto \mathcal{G}_n(x)$ is actually a polynomial which depends on the Bernoulli polynomial $B_n(X)$.

Proposition 4.1. *For all natural number n, we have*

$$\mathcal{G}_n(X) = (B_n(X) - B_n)^* = \sum_{k=0}^{n-1} \binom{n}{k} B_k X^k.$$

So $\mathcal{G}_n$ $(n \in \mathbb{N})$ is a polynomial with degree $\leq n-1$.

Proof. By definition of the Bernoulli polynomials, we have

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(X) \frac{t^n}{n!}.$$

Then, By substituting in the latter X by $\frac{1}{X}$ and t by Xt, we get

$$\frac{Xte^t}{e^{Xt}-1} = \sum_{n=0}^{+\infty} X^n B_n\left(\frac{1}{X}\right) \frac{t^n}{n!};$$

that is (since $\deg B_n = n$, $\forall n \in \mathbb{N}$)

$$\frac{Xt}{e^{Xt}-1}e^t \;=\; \sum_{n=0}^{+\infty} B_n^*(X)\frac{t^n}{n!}. \tag{4.1}$$

On the other hand, we have by definition of the Bernoulli numbers:

$$\frac{Xt}{e^{Xt}-1} \;=\; \sum_{n=0}^{+\infty} B_n\frac{(Xt)^n}{n!},$$

that is

$$\frac{Xt}{e^{Xt}-1} \;=\; \sum_{n=0}^{+\infty} B_n X^n\frac{t^n}{n!}. \tag{4.2}$$

By subtracting side to side (4.2) from (4.1), we finally obtain

$$\frac{Xt}{e^{Xt}-1}(e^t-1) \;=\; \sum_{n=0}^{+\infty}\left(B_n^*(X) - B_n X^n\right)\frac{t^n}{n!}.$$

Comparing this with the identity defining the $\mathcal{G}_n(X)$'s, we derive that for all $n \in \mathbb{N}$, we have

$$\mathcal{G}_n(X) \;=\; B_n^*(X) - B_n X^n \;=\; \left(B_n(X) - B_n\right)^*,$$

as required. The second equality of the proposition immediately follows from the well-known expression of the Bernoulli Polynomials in terms of the Bernoulli numbers, which is $B_n(X) = \sum_{k=0}^{n}\binom{n}{k}B_k X^{n-k}$ ($\forall n \in \mathbb{N}$). This completes the proof. □

Remark 4.2. *Since we know that $B_n = 0$ if and only if n is an odd integer ≥ 3, then from the formula of Proposition 4.1, we can precise the degree of the polynomial $\mathcal{G}_n$ ($n \in \mathbb{N}^*$). We have that:*

$$\deg \mathcal{G}_n \;=\; \begin{cases} n-1 & \text{if } n \text{ is odd or } n = 2 \\ n-2 & \text{if } n \text{ is even and } n \geq 4 \end{cases}.$$

Further, from Theorem 3.1, we derive the following corollary:

Corollary 4.3. *For any natural number n, the polynomial $\mathcal{G}_n(X)$ is integer-valued.*

To prove this corollary, we lean on the following well-known lemma (see, e.g., [2]):

Lemma 4.4. *Let $d \in \mathbb{N}$ and P be a polynomial of $\mathbb{Q}[[X]]$ with degree d. For P to be an integer-valued polynomial, it suffices that P takes integer values for $(d+1)$ consecutive integer values of X.*

Proof. Suppose that P takes integer values for some $(d+1)$ consecutive integer values of X, which are: $a, a+1, \dots, a+d$ ($a \in \mathbb{Z}$) and let us show that P is an integer-valued polynomial. Since $\deg P = d$ then we have that $\Delta^{d+1}P = 0$; that is

$$P(X+d+1) \;=\; \sum_{k=0}^{d}(-1)^{d-k}\binom{d+1}{k}P(X+k).$$

Using this identity, we immediately deduce by induction that:

$$P(x) \in \mathbb{Z} \qquad (\forall x \in \mathbb{Z}, x \geq a + d). \tag{4.3}$$

Next, if we take instead of $P(X)$ the polynomial $P(-X)$, which has the same degree with P and takes integer values for the $(d+1)$ consecutive integer values $-a-d$, $-a-d+1$, ..., $-a$ of X, we similarly obtain that:

$$P(-x) \in \mathbb{Z} \qquad (\forall x \in \mathbb{Z}, x \geq -a);$$

that is

$$P(x) \in \mathbb{Z} \qquad (\forall x \in \mathbb{Z}, x \leq a). \tag{4.4}$$

From (4.3) and (4.4), we conclude that $P(x) \in \mathbb{Z}$, $\forall x \in \mathbb{Z}$. Thus P is an integer-valued polynomial. The lemma is proved. □

Let us now prove Corollary 4.3:

Proof of Corollary 4.3. Let $n \in \mathbb{N}$ be fixed. Since for any integer $a \geq 2$, we have $\mathcal{G}_n(a) = G_{n,a} \in \mathbb{Z}$ (according to Theorem 3.1) then the polynomial $\mathcal{G}_n(X)$ takes integer values for an infinite number of consecutive integer values of X. It follows (according to Lemma 4.4) that $\mathcal{G}_n(X)$ is an integer-valued polynomial. This achieves the proof. □

Next, from Proposition 4.1 and Corollary 4.3, we derive the following curious result concerning the Bernoulli polynomials of odd degree.

Corollary 4.5. *For any odd integer $n \geq 3$, the reciprocal polynomial of the Bernoulli polynomial $B_n(X)$ is integer-valued.*

Proof. This is an immediate consequence of Proposition 4.1, Corollary 4.3 and the well-known fact that $B_n = 0$ for n odd, $n \geq 3$. □

We now turn to present another important property concerning the reciprocal polynomials of some particular type of integer-valued polynomials. For a given $n \in \mathbb{N}$, we define

$$\sigma_n(a) := 0^n + 1^n + \cdots + a^n \qquad (\forall a \in \mathbb{N}),$$

where we convention that $0^0 = 1$.

It has been known for a very long time that $\sigma_n(a)$ is polynomial on a with degree $(n+1)$, but a closed form of the polynomial in question was discovered for the first time by Jacob Bernoulli [1] and it is given by:

$$\sigma_n(a) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k a^{n+1-k} + a^n.$$

For a given $n \in \mathbb{N}$, let us define $\sigma_n(X)$ as the polynomial interpolating the sequence $(\sigma_n(a))_{a\in\mathbb{N}}$; that is

$$\sigma_n(X) = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k X^{n+1-k} + X^n. \tag{4.5}$$

For $n \in \mathbb{N}$, since the $\sigma_n(a)$'s ($a \in \mathbb{N}$) are obviously all integers then (according to Lemma 4.4) the polynomial $\sigma_n(X)$ is an integer-valued polynomial. But what about its reciprocal polynomial? The previous results permit us to obtain something in this direction. We have the following proposition:

Proposition 4.6. *For any natural number n, the polynomial $(n+1)\sigma_n^*(X)$ is integer-valued.*

Proof. Let $n \in \mathbb{N}$ be fixed. According to (4.5), we have

$$\begin{aligned}\sigma_n^*(X) &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k X^k + X \\ &= \frac{1}{n+1}\mathcal{G}_{n+1}(X) + X \qquad \text{(according to Proposition 4.1).}\end{aligned}$$

Hence

$$(n+1)\sigma_n^*(X) = \mathcal{G}_{n+1}(X) + (n+1)X.$$

Since $\mathcal{G}_{n+1}(X)$ is integer-valued (according to Corollary 4.3), the last equality shows that also $(n+1)\sigma_n^*(X)$ is integer-valued (as the sum of two integer-valued polynomials). The proposition is proved. □

Two important open problems

1.— Since the polynomials $\mathcal{G}_n(X)$ are integer-valued (according to Corollary 4.3) then they admit representations as linear combinations, with integer coefficients, of the polynomials $\binom{X}{k}$ ($k \in \mathbb{N}$). Precisely, there exist integers $a_{n,k}$ ($n \in \mathbb{N}^*$, $k \in \mathbb{N}$, $0 \leq k < n$) for which we have

$$\mathcal{G}_n(X) = a_{n,0}\binom{X}{0} + a_{n,1}\binom{X}{1} + \cdots + a_{n,n-1}\binom{X}{n-1} \qquad (\forall n \in \mathbb{N}^*).$$

The $a_{n,k}$'s can be calculated for example by using the Newton interpolation formula:

$$P(X) = \sum_{k=0}^{\deg P}\left(\Delta^k P\right)(0)\binom{X}{k} \qquad (\forall P \in \mathbb{Q}[X]).$$

So, we have that:

$$a_{n,k} = \left(\Delta^k \mathcal{G}_n\right)(0) \qquad (\forall n \in \mathbb{N}^*, \forall k \in \mathbb{N}, 0 \leq k < n).$$

If we arrange those integers $a_{n,k}$ ($0 \leq k < n$) in a triangle array in which each $a_{n,k}$ is the entry in the n^{th} row and k^{th} column, we obtain (after calculation) the following configuration:

$n = 1$	1						
$n = 2$	1	-1					
$n = 3$	1	-1	1				
$n = 4$	1	-1	2				
$n = 5$	1	-1	1	-6	-4		
$n = 6$	1	-1	-2	-18	-12		
$n = 7$	1	-1	1	48	232	300	120
$n = 8$	1	-1	18	276	984	1200	480

Table 1: The triangle of the $a_{n,k}$'s for $0 \leq k < n \leq 8$

Note that in this triangle, we have omitted the integers $a_{n,n-1}$ for the even n's (since they are zero).
The interesting problem we pose here consists to find a simple and practical rule to construct the above triangle step by step.

2.— For a polynomial $P \in \mathrm{Int}(\mathbb{Z})$, we don't have in general $(\deg P) \cdot P^* \in \mathrm{Int}(\mathbb{Z})$ (indeed, the polynomial $\binom{X}{3} = \frac{X(X-1)(X-2)}{6}$ provides a counterexample); however, the polynomials $\sigma_n(X)$ ($n \in \mathbb{N}$) satisfy this property (according to Proposition 4.6). So, it is interesting to study for which category of integer-valued polynomials, the above property is satisfied.

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