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Arithmetic properties of the Genocchi numbers and their generalization

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Abstract

This note is devoted to establish some new arithmetic properties of the generalized Genocchi numbers $G_{n,a}$ ($n \in \mathbb{N}$, $a \geq 2$). The resulting properties for the usual Genocchi numbers $G_n = G_{n,2}$ are then derived. We show for example that for any even positive integer n, the Genocchi number G_n is a multiple of the odd part of n.

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1 Introduction and Notations

Throughout this note, we let $\mathbb{N}^*$ denote the set of positive integers. For a given prime number p, we let ϑ_p denote the usual p-adic valuation. For a given rational number r, we let $\mathrm{num}(r)$ and $\mathrm{den}(r)$ respectively denote the numerator and the denominator of r; precisely, if d is the smallest positive integer satisfying $dr \in \mathbb{Z}$ then we have: $\mathrm{den}(r) := d$ and $\mathrm{num}(r) := dr$. Next, for given positive integers a and n, we let $\pi_a(n)$ denote the greatest positive divisor of n which is coprime with a. Then, it is immediate that:

$$\pi_a(n) = \prod_{\substack{p \text{ prime} \\ p \nmid a}} p^{\vartheta_p(n)}.$$

In the particular case $a = 2$, $\pi_a(n)$ is called *the odd part* of n; it is the greatest odd positive divisor of n.

Additionally, we need to extend the congruences in $\mathbb{Z}$ to $\mathbb{Q}$ as follows:

For $a, b \in \mathbb{Q}$ and $n \in \mathbb{N}^*$, we write $a \equiv b \pmod{n}$ if the numerator of the rational number $(a - b)$ is a multiple of n.

In the same context, an equivalent meaning of the congruence $a \equiv b \pmod{n}$ consists to say that for any prime divisor p of n, we have $\vartheta_p(a - b) \geq \vartheta_p(n)$. We can easily check that some properties (but not all) of the congruences in $\mathbb{Z}$ become valid in $\mathbb{Q}$. For example, we can sum side to side several congruences in $\mathbb{Q}$ with a same modulus. Also, if $a, b, c \in \mathbb{Q}$ and $n \in \mathbb{N}^*$ such that $\mathrm{den}(c)$ is coprime with n then we have:

$$a \equiv b \pmod{n} \implies ac \equiv bc \pmod{n}.$$

However, we cannot multiply (side to side) several congruences in $\mathbb{Q}$ with a same modulus (contrary to the congruences in $\mathbb{Z}$).

Further, we denote by $\mathbb{Q}[[t]]$ the ring of formal power series in t with coefficients in $\mathbb{Q}$. If an element f of $\mathbb{Q}[[t]]$ is represented as

$$f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$$

(where $a_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$), we call the a_n's *the differential coefficients* of f (because each a_n is the n^{th} derivative of f at 0). If the a_n's are all integers, we say that f is an *IDC-series* (IDC abbreviates the expression "with Integral Differential Coefficients"). Many usual functions are IDC-series; we can cite for example the functions $x \mapsto e^x$, $x \mapsto \sin x$, $x \mapsto \cos x$, $x \mapsto \ln(1 + x)$, and so on. We can easily see that the sum and the product of two IDC-series become an IDC-series. Other important properties of the IDC-series are given by the author [3] who used them to establish his generalization of the Genocchi theorem (see below). One of those properties is given by the following proposition:

Proposition 1.1 ([3, Corollary 2.2]). *Let $f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$ be an IDC-series with $a_0 \neq 0$. Then the formal power series $\frac{a_0}{f(a_0 t)}$ is also an IDC-series.*

Showing that a given function is an IDC-series is, in general, not easy. For example, the statement that the function $t \mapsto \frac{2t}{e^t+1}$ is an IDC-series constitute a profound theorem due to Genocchi [5]. The differential coefficients of the last function (which thus are all integers) are called *the Genocchi numbers* and denoted by G_n ($n \in \mathbb{N}$); so we have:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!}.$$

For a study of the Genocchi numbers, the reader can consult [1, 2, 3, 4, 5, 8].

Very recently, the author [3] has generalized the Genocchi numbers by considering, for a given integer $a \geq 2$, the function $t \mapsto \frac{at}{e^{(a-1)t}+e^{(a-2)t}+\cdots+e^t+1}$ and its expansion into a power series:

$$\frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}.$$

So, for $a = 2$, we simply obtain the usual Genocchi numbers; that is $G_{n,2} = G_n$ ($\forall n \in \mathbb{N}$). The main result of [3] then states that the $G_{n,a}$'s are all integers, which generalizes the Genocchi theorem. The Genocchi numbers as well as their generalization are ultimately related to the Bernoulli numbers $(B_n)_{n\in\mathbb{N}}$, which can be defined by the exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} B_n \frac{t^n}{n!}$$

(see e.g., [1, 7]). Among others, we have the following connection between the generalized Genocchi numbers and the Bernoulli numbers:

Proposition 1.2 ([3, Proposition 4.1])**.** *For any positive integers a and n, with $a \geq 2$, we have:*

$$G_{n,a} = \sum_{k=0}^{n-1} \binom{n}{k} B_k a^k.$$

Furthermore, the Bernoulli numbers have many arithmetic properties. The most famous is certainly that given by the von Staudt-Clausen theorem (see e.g., [6, 7]), recalled below:

Theorem 1.3 (von Staudt-Clausen)**.** *For any even positive integer n, we have that:*

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)\mid n}} \frac{1}{p} \in \mathbb{Z}.$$

Because it is known that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_n = 0$ for any odd integer $n \geq 3$, we can derive from the von Staudt-Clausen theorem the following immediate corollary which is useful for our purpose:

Corollary 1.4. *For any natural number n and any prime number p, we have that:*

$$\vartheta_p(B_n) \geq -1.$$

The purpose of this note consists to establish arithmetic properties of the numbers $G_{n,a}$ (in addition to their integrality, previously proven in [3]). Precisely, we obtain for the $G_{n,a}$'s a nontrivial simple divisor and a nontrivial congruence modulo a.

2 The results and the proofs

We begin by establishing nontrivial divisors for the generalized Genocchi numbers.

Theorem 2.1. *Let $a \geq 2$ be an integer. Then for all positive integer n, the integer $G_{n,a}$ is a multiple of the integer $\pi_a(n)$.*

Proof. By applying Proposition 1.1 to the IDC-series $t \mapsto e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1$, we find that the function $t \overset{f}{\mapsto} \frac{a}{e^{(a-1)at}+e^{(a-2)at}+\cdots+e^{at}+1}$ is an IDC-series. Now, let us explicitly express the differential coefficients of f in terms of the generalized Genocchi numbers. We have:

$$\begin{aligned}
\frac{a}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} &= \frac{1}{t} \cdot \frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} \\
&= \frac{1}{t} \sum_{n=1}^{+\infty} G_{n,a} \frac{t^n}{n!} \\
&= \sum_{n=1}^{+\infty} \frac{G_{n,a}}{n} \cdot \frac{t^{n-1}}{(n-1)!}.
\end{aligned}$$

Then, by substituting t by at, we get

$$f(t) = \sum_{n=1}^{+\infty} \frac{a^{n-1} G_{n,a}}{n} \cdot \frac{t^{n-1}}{(n-1)!}.$$

So, since f is an IDC-series, we derive that $\frac{a^{n-1}G_{n,a}}{n} \in \mathbb{Z}$ $(\forall n \in \mathbb{N}^*)$; that is

$$n \mid a^{n-1} G_{n,a} \qquad (\forall n \in \mathbb{N}^*).$$

Next, since $\pi_a(n)$ is a divisor of n which is coprime with a, we derive that $\pi_a(n) \mid a^{n-1}G_{n,a}$ and $\pi_a(n)$ is coprime with a^{n-1}. Then, by Gauss's lemma, we conclude that $\pi_a(n)$ divides $G_{n,a}$, as required. □

By taking $a = 2$ in Theorem 2.1, we immediately deduce the following corollary which concerns the usual Genocchi numbers:

Corollary 2.2. *For all positive integer n, the Genocchi number G_n is a multiple of the odd part of n.* □

We now turn to study the remainders of the generalized Genocchi numbers $G_{n,a}$ modulo a. The main result in this direction is the following:

Theorem 2.3. *For any integers a and n, both greater than 1, we have*

$$G_{n,a} \equiv 1 - \frac{n}{2} a \pmod{a}.$$

Proof. Let a and n be two integers, both greater than 1. According to Proposition 1.2, we have that:

$$\begin{aligned}
G_{n,a} &= \sum_{k=0}^{n-1} \binom{n}{k} B_k a^k \\
&= 1 - \frac{n}{2} a + \sum_{k=2}^{n-1} \binom{n}{k} B_k a^k. \qquad (2.1)
\end{aligned}$$

Next, by using Corollary 1.4, we have for any $k \in \{2, 3, \ldots, n-1\}$ and any prime divisor p of a (so $\vartheta_p(a) \geq 1$):

$$\begin{aligned} \vartheta_p\left(B_k a^k\right) &= \vartheta_p\left(B_k\right) + (k-1)\vartheta_p(a) + \vartheta_p(a) \\ &\geq -1 + (k-1) + \vartheta_p(a) \\ &= k - 2 + \vartheta_p(a) \\ &\geq \vartheta_p(a), \end{aligned}$$

showing that $B_k a^k \equiv 0 \pmod a$ $(\forall k \in \{2, 3, \ldots, n-1\})$. By substituting these congruences into (2.1), we conclude to the required congruence $G_{n,a} \equiv 1 - \frac{n}{2}a \pmod a$. $\square$

From Theorem 2.3, we immediately derive the following corollary:

Corollary 2.4. *Let $a \geq 2$ be an integer.*

- *If a is odd then we have for any positive integer n:*

$$G_{n,a} \equiv 1 \pmod a.$$

- *If a is even then we have for any integer $n \geq 2$:*

$$G_{n,a} \equiv \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 + \frac{a}{2} & \text{if } n \text{ is odd} \end{cases} \pmod a. \qquad \square$$

Remark 2.5. *By applying Corollary 2.4 for $a = 2$, we obtain the well-known result stating that the usual Genocchi numbers of even positive orders are all odd.*

We conclude this note by the following corollary which is an immediate consequence of Corollary 2.4 above.

Corollary 2.6. *For any integers a and n, both greater than 1, we have:*

$$\gcd\left(G_{n,a}, a\right) \in \{1, 2\}.$$

Besides, the equality $\gcd(G_{n,a}, a) = 2$ holds if and only if $a \equiv 2 \pmod 4$ and n is odd. $\square$

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