AN EXPLICIT FORMULA GENERATING THE NON-FIBONACCI NUMBERS

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Abstract. We show among others that the formula:

$$\left\lfloor n + \log_\Phi\left\{\sqrt{5}\left(\log_\Phi(\sqrt{5}n) + n\right) - 5 + \frac{3}{n}\right\} - 2\right\rfloor \quad (n \geq 2),$$

(where Φ denotes the golden ratio and $\lfloor\rfloor$ denotes the integer part) generates the non-Fibonacci numbers.

1. Introduction and main result

Two sequences of natural numbers are said to be complementary if they are disjoint and their union is the entire set $\mathbb{N}$ of nonnegative integers. Given a sequence of nonnegative integers, it is an important problem to find an explicit formula for the sequence complementing them in $\mathbb{N}$. For example, it is well known that the sequence complementing the perfect square positive integers is generated by the formula $\lfloor n + \sqrt{n} + \frac{1}{2}\rfloor$ $(n \in \mathbb{N})$ and the sequence complementing the triangular numbers (i.e., the integers having the form $\frac{n(n+1)}{2}$, $n \in \mathbb{N}$) is generated by the formula $\lfloor n + \sqrt{2n} + \frac{1}{2}\rfloor$ $(n \geq 1)$. On this topic we can consult the article [5] of Lambekand and Moser. For a brief introduction, we can consult the book [4] of Honsberger or the chapter 1 of the book [3] of the same author.

In this paper, we establish a general theorem which gives the formula generating the complement (in $\mathbb{N}$) of a given sequence of integers. Then we apply it to obtain the complementary sequences of some types of sequences including the Fibonacci sequence. So we obtain an explicit formula for the n^{th} non-fibonacci number. Actually, Gould [2] already obtained an approximate formula for the n^{th} non-Fibonacci number (noted g_n). His formula is:

$$g_n = n + F(n + F(n + F(n))),$$

where $F(n) = \lfloor \log_\Phi n + \frac{1}{2}\log_\Phi 5 - 1\rfloor$. But the inconvenient of Gould's formula is that it is quite complicated and the purpose of this paper is to obtain an easy formula depending only on n. In addition, our approach is somewhat different from that of Gould. Our main result is the following:

Theorem 1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence of integers and $\varphi : [0, +\infty[\rightarrow \mathbb{R}$ be a continued function which increases and tends to $+\infty$ when x tends to $+\infty$. Suppose that φ satisfies for all $n \in \mathbb{N}$:*

$$u_n - n < \varphi(n) \leq u_n - n + 1 \qquad (I)$$

Then the formula $(\lfloor n + \varphi^{-1}(n)\rfloor)_{n\geq u_0+1}$ generates the complement of $\{u_n, n \in \mathbb{N}\}$ in $[u_0, +\infty[\cap\mathbb{Z}$.

Proof.
First remark that the hypothesis (I) of the theorem gives $\varphi(0) \leq u_0 + 1$. Consequently, since φ is continuous, increasing and tends to $+\infty$ when x tends to $+\infty$, the set of arrival of φ contains the interval $[u_0 + 1, +\infty[$. It follows that φ^{-1} is defined at least in the interval $[u_0 + 1, +\infty[$.

Also, because $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of integers, we have for all $n \in \mathbb{N}$: $u_n \geq u_0 + n$. In particular φ^{-1} can be applied to all real number greater than or equal to $u_n - n + 1$ ($n \in \mathbb{N}$).

• Now, let $N \in [u_0, +\infty[\cap \mathbb{Z}$ which is not a term of $(u_n)_n$ and let us show that N is a term of the sequence $(\lfloor k + \varphi^{-1}(k) \rfloor)_{k \geq u_0+1}$. Since $N \geq u_0$ then N lies between two consecutive terms of $(u_n)_n$. Let $n \in \mathbb{N}$ such that:

$$u_n < N < u_{n+1}.$$

since N is an integer, we have also:

$$u_n + 1 \leq N \leq u_{n+1} - 1.$$

Hence

$$u_n - n + 1 \leq N - n \leq u_{n+1} - (n+1).$$

By applying φ^{-1} (which is increasing because φ is increasing), it follows that:

$$\varphi^{-1}(u_n - n + 1) \leq \varphi^{-1}(N - n) \leq \varphi^{-1}(u_{n+1} - (n+1)).$$

But according to the hypothesis (I), we have $\varphi^{-1}(u_{n+1} - (n+1)) < n+1$ and $\varphi^{-1}(u_n - n + 1) \geq n$. So

$$n \leq \varphi^{-1}(N - n) < n + 1,$$

which implies that:

$$\lfloor \varphi^{-1}(N - n) \rfloor = n.$$

Finally, we conclude that:

$$N = (N - n) + n = (N - n) + \lfloor \varphi^{-1}(N - n) \rfloor = \lfloor (N - n) + \varphi^{-1}(N - n) \rfloor,$$

which implies that N is generated by the formula $\lfloor k + \varphi^{-1}(k) \rfloor$ ($k \geq u_0 + 1$).

• Conversely, let N be a term of the sequence $(\lfloor k + \varphi^{-1}(k) \rfloor)_{k \geq u_0+1}$ and let us show that N is not a term of $(u_k)_{k\in\mathbb{N}}$.

Let $n \geq u_0 + 1$ be fixed such that $N = \lfloor n + \varphi^{-1}(n) \rfloor$. So we have:

$$N \leq n + \varphi^{-1}(n) < N + 1.$$

By subtracting n and then applying φ (which is increasing), we get:

$$\varphi(N - n) \leq n < \varphi(N - n + 1).$$

But according to the hypothesis (I), we have: $\varphi(N - n + 1) \leq u_{N-n+1} - (N - n + 1) + 1 = u_{N-n+1} - N + n$ and $\varphi(N - n) > u_{N-n} - (N - n) = u_{N-n} - N + n$. Using this, we get:

$$u_{N-n} - N + n < n < u_{N-n+1} - N + n,$$

which is equivalent to:

$$u_{N-n} < N < u_{N-n+1}.$$

So N lies between two consecutive terms of $(u_k)_k$. Hence N cannot be a term of $(u_k)_k$.
This completes the proof of the theorem. ■

2. Applications

2.1. The Complement of the sequence $(n^r)_{n\in\mathbb{N}}$, $r \in \mathbb{N}, r \geq 2$. We have the following:

Corollary 2.1. *Let $r \geq 2$ be an integer. The formula $\left(\left\lfloor n + \sqrt[r]{n + \sqrt[r]{n}}\right\rfloor\right)_{n\geq 1}$ generates the positive integers which are not r^{th} powers.*

Proof.
We apply Theorem 1.1 for $u_n = n^r$ $(n \in \mathbb{N})$ and $\psi(x) := \varphi^{-1}(x) = \sqrt[r]{x + \sqrt[r]{x}}$ $(x \in [0, +\infty[)$ which is a continuous and increasing function and tends to $+\infty$ when x tends to $+\infty$. To verify the hypothesis (I) of Theorem 1.1, it is equivalent to verify that

$$\psi(n^r - n) < n \text{ and } \psi(n^r - n + 1) \geq n \ (\forall n \in \mathbb{N}).$$

In odrer to show that $\psi(n^r - n) < n$, it suffices to bound from above $\sqrt[r]{n^r - n}$ by n and in order to show that $\psi(n^r - n + 1) \geq n$ it suffices to bound from bellow $\sqrt[r]{n^r - n + 1}$ by $(n - 1)$. The corollary follows. ■

Remark. For the sequences of perfect squares and perfect cubes, we have other formulas more sample than the previous one complementing them. Indeed, using Theorem 1.1, we can show that the formula $(\lfloor n + \sqrt{n} + \frac{1}{2}\rfloor)_{n\geq 1}$ generates the positive integers which are not perfect squares and the formula $\left(\left\lfloor n + \sqrt[3]{n} + \frac{1}{3\sqrt[3]{n+1}}\right\rfloor\right)_{n\geq 1}$ generates the positive integers which are not perfect cubes.

2.2. The complement of the sequence $(a^n)_{n\in\mathbb{N}}$, $a \in \mathbb{N}$, $a \geq 2$. We have the following:

Corollary 2.2. *Let $a \geq 2$ be an integer. The formula $\lfloor n + \log_a(n + \log_a(n))\rfloor$ $(n \geq 1)$ generates the positive integers which are not powers of a.*

Proof.
We apply Theorem 1.1 for $u_n = a^n$ $(n \in \mathbb{N})$ and $\psi(x) := \varphi^{-1}(x) = \log_a(x + \log_a(x))$ $(x \in [1, +\infty[)$ which is a continuous and increasing function and tends to $+\infty$ when x tends to $+\infty$. To verify the hypothesis (I) of Theorem 1.1, we have to verify that:

$$\psi(a^n - a) < n \text{ and } \psi(a^n - n + 1) \geq n \ (\forall n \in \mathbb{N}).$$

Those inequalities easily follow from the trivial upper bound $\log_a(a^n - n) < n$ and the trivial lower bound $\log_a(a^n - n + 1) > n - 1$. The corollary follows. ■

2.3. The complement of the Fibonacci sequence. The Fibonacci sequence, noted $(F_n)_{n\in\mathbb{N}}$, is defined by:

$$\begin{cases} F_0 = 0 \ , \ F_1 = 1 \\ F_{n+2} = F_n + F_{n+1} \ (\forall n \in \mathbb{N}) \end{cases}.$$

The Fibonacci numbers are simply the terms of $(F_n)_n$. The complement of $(F_n)_n$ is given by the following:

Corollary 2.3. *The formula*

$$\left\lfloor n + \log_\Phi\left\{\sqrt{5}\left(\log_\Phi(\sqrt{5}n) + n\right) - 5 + \frac{3}{n}\right\} - 2\right\rfloor \quad (n \geq 2)$$

generates the numbers which are not Fibonacci numbers.

Proof.
We apply Theorem 1.1 for $u_n = F_{n+2}$ $(n \in \mathbb{N})$ and

$$\psi(x) := \varphi^{-1}(x) = \log_\Phi \left\{ \sqrt{5}\left(\log_\Phi(\sqrt{5}x) + x\right) - 5 + \frac{3}{x} \right\} - 2,$$

which is a continuous and increasing function on $[2, +\infty[$ and tends to $+\infty$ when x tends to $+\infty$.
To verify the hypothesis (I) of Theorem 1.1, we have to verify that:

$$\psi(F_{n+2} - n) < n \text{ and } \psi(F_{n+2} - n + 1) \geq n \ (\forall n \in \mathbb{N}).$$

To do so, we verify those inequalities for the small values of n $(n \leq 10)$ and we use Binet's formula (see for example [3], chapter 8):

$$F_n = \frac{1}{\sqrt{5}}\left(\Phi^n - \overline{\Phi}^n\right)$$

(where $\overline{\Phi} = \frac{1-\sqrt{5}}{2} = -\frac{1}{\Phi}$) to verify them for the large values of n $(n > 10)$. Let us prove the above inequalities for the large values of n. Using Binet's formula, the calculations give:

$$\sqrt{5}\left(\log_\Phi(\sqrt{5}(F_{n+2} - n)) + F_{n+2} - n\right) - 5 + \frac{3}{F_{n+2} - n} = \Phi^{n+2} - \overline{\Phi}^{n+2}$$
$$+ 2\sqrt{5} - 5 + \frac{3}{F_{n+2} - n} + \sqrt{5}\log_\Phi\left\{1 - (-\overline{\Phi}^2)^{n+2} - \sqrt{5}n(-\overline{\Phi})^{n+2}\right\}.$$

Because $2\sqrt{5} - 5 < 0$ and the quantity

$$-\overline{\Phi}^{n+2} + \frac{3}{F_{n+2} - n} + \sqrt{5}\log_\Phi\left\{1 - (-\overline{\Phi}^2)^{n+2} - \sqrt{5}n(-\overline{\Phi})^{n+2}\right\}$$

tends to 0 as n tends to infinity, we have:

$$\sqrt{5}\left(\log_\Phi(\sqrt{5}(F_{n+2} - n)) + F_{n+2} - n\right) - 5 + \frac{3}{F_{n+2} - n} < \Phi^{n+2}$$

for n sufficiently large (in practice $n > 10$ suffices). This gives $\psi(F_{n+2} - n) < n$, as required.
Similarly, using Binet's Formula, the calculations give:

$$\sqrt{5}\left(\log_\Phi(\sqrt{5}(F_{n+2} - n + 1)) + F_{n+2} - n + 1\right) - 5 + \frac{3}{F_{n+2} - n + 1} =$$
$$\Phi^{n+2} - \overline{\Phi}^{n+2} + 3\sqrt{5} - 5 + \frac{3}{F_{n+2} - n + 1} + \sqrt{5}\log_\Phi\left\{1 - (-\overline{\Phi}^2)^{n+2} - \sqrt{5}(n-1)(-\overline{\Phi})^{n+2}\right\}.$$

Because $3\sqrt{5} - 5 > 0$ and the quantity

$$-\overline{\Phi}^{n+2} + \frac{3}{F_{n+2} - n + 1} + \sqrt{5}\log_\Phi\left\{1 - (-\overline{\Phi}^2)^{n+2} - \sqrt{5}(n-1)(-\overline{\Phi})^{n+2}\right\}$$

tends to 0 as n tends to infinity then for n sufficiently large $(n > 10$ suffices) we have:

$$\sqrt{5}\left(\log_\Phi(\sqrt{5}(F_{n+2} - n + 1)) + F_{n+2} - n + 1\right) - 5 + \frac{3}{F_{n+2} - n + 1} > \Phi^{n+2}.$$

This gives $\psi(F_{n+2} - n + 1) > n$ (for $n > 10$). The proof is complete. ■

Remark on the sequences with several indices

We don't know how to generalize Theorem 1.1 for the sequences of several indices although there exist some theorems of complement of sequences with several indices. The more famous is perhaps Legendre's theorem (see e.g. [1]) which states that the sequence with three indices $(n^2 + m^2 + k^2)_{n,m,k \in \mathbb{N}}$ has for complement (in $\mathbb{N}$) the sequence with two indices $(4^h(8\ell + 7))_{h,\ell \in \mathbb{N}}$.

Note also that if we are able to generalize Theorem 1.1 for sequences with two indices then we can obtain a formula generating prime numbers, because it is obvious that the sample formula $((n+2)(m+2))_{n,m \in \mathbb{N}}$ generates the composite numbers.

References

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