J. Integer Sequences,
23 (2020), Article 20.6.6.

Nontrivial Effective Lower Bounds for the Least Common Multiple of Some Quadratic Sequences

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Abstract

This paper is devoted to studying the numbers

$$L_{c,m,n} := \operatorname{lcm}\left(m^2+c, (m+1)^2+c,\ \dots, n^2+c\right),$$

where c, m, n are positive integers such that $m \leq n$. More precisely, we determine a nontrivial rational divisor of $L_{c,m,n}$ and then we derive (as consequences) some nontrivial lower bounds for $L_{c,m,n}$. Our approach (focusing on commutative algebra) is new and different from those using previously by Farhi, Oon, and Hong.

1 Introduction and Notation

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers. For $t \in \mathbb{R}$, we let $\lfloor t \rfloor$ and $\lceil t \rceil$ respectively denote the floor and the ceiling functions. We say that an integer a is a multiple of a non-zero rational number r (or equivalently, r is a divisor of a) if the quotient a/r is an integer. If m, n, c are positive integers such that $m \leq n$, we set $L_{c,m,n} := \operatorname{lcm}\{m^2+c, (m+1)^2+c, \ldots, n^2+c\}$. For a given polynomial $P \in \mathbb{C}[X]$, we let $\overline{P}$ denote the polynomial conjugate of P in $\mathbb{C}[X]$, that is the polynomial we get by replacing each coefficient of P with its complex conjugate. It is well-known that the conjugation of polynomials in $\mathbb{C}[X]$ is compatible with addition and multiplication, in the sense that for every $P, Q \in \mathbb{C}[X]$, we have $\overline{P+Q} = \overline{P} + \overline{Q}$ and $\overline{P \cdot Q} = \overline{P} \cdot \overline{Q}$. Further, we let I, E_h ($h \in \mathbb{R}$), and Δ denote the linear operators on $\mathbb{C}[X]$ which respectively represent the identity, the shift operator with step h ($E_h P(X) = P(X+h)$, $\forall P \in \mathbb{C}[X]$), and the forward difference ($\Delta P(X) = P(X+1) - P(X)$, $\forall P \in \mathbb{C}[X]$). For $n \in \mathbb{N}$, the expression of Δ^n in terms of the E_h's is easily obtained from the binomial formula, as follows

$$\Delta^n = (E_1 - I)^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} E_1^m = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} E_m. \tag{1}$$

For falling factorial powers, we use Knuth's notation:

$$X^{\underline{n}} := X(X-1)(X-2) \cdots (X-n+1) \qquad (\forall n \in \mathbb{N}).$$

The study of the least common multiple of the first n consecutive positive integers ($n \in \mathbb{N}^*$) began with Chebyshev's work [2] in his attempts to prove the prime number theorem. The latter showed that the prime number theorem is equivalent to stating that $\log \operatorname{lcm}(1, 2, \ldots, n) \sim_{+\infty} n$. More recently, many authors are interested in the effective estimates of the least common multiple of consecutive terms of some integer sequences. In 1972, Hanson [6] showed (by using the expansion of the number 1 in Sylvester series) that $\operatorname{lcm}(1, 2, \ldots, n) \leq 3^n$ ($\forall n \in \mathbb{N}^*$). In 1982, investigating the integral $\int_0^1 x^n (1-x)^n \mathrm{d}x$, Nair [11] gave a simple proof that $\operatorname{lcm}(1, 2, \ldots, n) \geq 2^n$ ($\forall n \geq 7$). Later, the second author [5] obtained nontrivial lower bounds for the least common multiple of consecutive terms in an arithmetic progression. In particular, he proved that for any $u_0, r, n \in \mathbb{N}^*$ such that $\gcd(u_0, r) = 1$, we have

$$\operatorname{lcm}(u_0, u_0 + r, \ldots, u_0 + nr) \geq u_0 (r+1)^{n-1}, \tag{2}$$

and conjectured that the exponent $(n-1)$ appearing in the right-hand side of (2) can be replaced by n, which is the optimal exponent that can be obtained. That conjecture was confirmed by Hong and Feng [7]. Furthermore, several authors obtained improvements of (2) for n sufficiently large in terms of u_0 and r (see, e.g., [7, 8, 10]). The second author [5] also obtained nontrivial lower bounds for the least common multiple of some quadratic sequences. In particular, he proved that for any positive integer n, we have

$$\operatorname{lcm}\left(1^2 + 1, 2^2 + 1, \ldots, n^2 + 1\right) \geq 0.32(1.442)^n. \tag{3}$$

In 2013, Oon [12] managed to improve (3) by proving that for any positive integers c and n, we have

$$\operatorname{lcm}\left(1^2 + c, 2^2 + c, \ldots, n^2 + c\right) \geq 2^n. \tag{4}$$

Actually, we have something a little stronger:

Theorem 1 (Oon [12])**.** *Let c, n, m be positive integers such that $m \leq \lceil \frac{n}{2} \rceil$. Then we have*

$$L_{c,m,n} \geq 2^n.$$

Later, Hong et al. [9] managed to generalize Theorem 1 for polynomial sequences $(f(n))_{n\geq 1}$, with $f \in \mathbb{Z}[X]$ and the coefficients of f are all nonnegative. In another direction, various asymptotic estimates have been obtained by several authors. For example, Bateman et al. [1] proved that for any $h, k \in \mathbb{Z}$ with $k > 0$, $h + k > 0$, and $\gcd(h, k) = 1$, we have

$$\log \operatorname{lcm}\{h + k, h + 2k, \dots, h + nk\} \ \sim_{+\infty} \ \left(\frac{k}{\varphi(k)} \sum_{\substack{1\leq m\leq k \\ \gcd(m,k)=1}} \frac{1}{m} \right) n, \tag{5}$$

where φ denotes the Euler totient function. Another asymptotic estimate a little harder to prove is due to Cilleruelo [3] and states that for every irreducible quadratic polynomial $f \in \mathbb{Z}[X]$, we have

$$\log \operatorname{lcm}\{f(1), \dots, f(n)\} = n \log n + Bn + o(n), \tag{6}$$

where B is a constant depending on f.

In this paper, we use arguments of commutative algebra and complex analysis to find a nontrivial rational divisor of $L_{c,m,n}$ ($c, m, n \in \mathbb{N}^*$). As a consequence, we derive some new nontrivial lower bounds for $L_{c,m,n}$. The rest of the paper is organized in five parts (subsections). In the first part, we give an algebraic lemma which allows us, on the one hand to re-demonstrate Theorem 1 of Oon by an easy and purely algebraic method, and on the other hand to reformulate the problem of bounding the number $L_{c,m,n}$ from below. In this reformulation, we are led to introduce a vital arithmetic function, noted h_c, whose multiple provides a divisor for $L_{c,m,n}$. In the next two parts, we study the arithmetic function h_c and we find for h_c a simple multiple. In the fourth part, we use the obtained multiple of h_c to deduce a nontrivial divisor for $L_{c,m,n}$. Our new nontrivial lower bounds for $L_{c,m,n}$ then follow from this divisor. We conclude the paper with a short last part which presents comparisons between our results and those of Oon (cf. Theorem 1).

2 The results and the proofs

2.1 An algebraic method

Although the method used by Oon [12] to obtain his result (i.e., Theorem 1) is analytic, the ingredients for its success are algebraic in depth, as we will show it below by applying the following fundamental algebraic lemma:

Lemma 2. *Let $\mathcal{A}$ be an integral domain and n be a positive integer. Let also $u_0, u_1, \dots, u_n$, a, b be elements of $\mathcal{A}$. Suppose that a and b satisfy the following conditions:*

1. *Each of the elements* $u_0, u_1, \dots, u_n$ *of* $\mathcal{A}$ *divides* a.

2. *Each of the elements* $\prod_{\substack{0 \leq j \leq n \\ j \neq i}} (u_i - u_j)$ $(i = 0, 1, \dots, n)$ *of* $\mathcal{A}$ *divides* b.

Then the product ab *is a multiple of the product* $u_0 u_1 \cdots u_n$.

Proof. If the elements $u_0, u_1, \dots, u_n$ of $\mathcal{A}$ are not pairwise distinct, the result of the lemma is trivial, since by its second condition we have $b = 0_{\mathcal{A}}$. Suppose in what follows that the u_i's $(i = 0, 1, \dots, n)$ are pairwise distinct. We use the well-known result that if a polynomial in one indeterminate, with coefficients in an integral domain, has a number of zeros (in that domain) greater than its degree then it is zero. Since a is a multiple of each of the elements $u_0, u_1, \dots, u_n$ of $\mathcal{A}$, there exist $k_0, k_1, \dots, k_n \in \mathcal{A}$ such that:

$$a = k_0 u_0 = k_1 u_1 = \cdots = k_n u_n. \tag{7}$$

Similarly, since b is a multiple of each of the elements $\prod_{\substack{0 \leq j \leq n \\ j \neq i}} (u_i - u_j)$ $(i = 0, 1, \dots, n)$, then there exist $\ell_0, \ell_1, \dots, \ell_n \in \mathcal{A}$ such that:

$$b = \ell_i \prod_{\substack{0 \leq j \leq n \\ j \neq i}} (u_i - u_j) \quad (\forall i \in \{0, 1, \dots, n\}). \tag{8}$$

Now, consider the following polynomial of $\mathcal{A}[X]$:

$$P(X) := \sum_{i=0}^{n} \left(\ell_i \prod_{\substack{0 \leq j \leq n \\ j \neq i}} (X - u_j) \right) - b.$$

We have $\deg P \leq n$. On the other hand, we have (according to (8))

$$P(u_i) = 0 \quad (\forall i \in \{0, 1, \dots, n\}),$$

showing that the number of zeros of P in $\mathcal{A}$ is greater than its degree. So, according to the elementary result of commutative algebra announced above, the polynomial P is zero. In particular, we have $P(0) = 0$; that is

$$b = (-1)^n \sum_{i=0}^{n} \ell_i \left(\prod_{\substack{0 \leq j \leq n \\ j \neq i}} u_j \right).$$

By multiplying the two sides of this last equality by a, we get (according to (7))

$$\begin{aligned} ab &= (-1)^n \sum_{i=0}^{n} \ell_i a \left(\prod_{\substack{0 \le j \le n \\ j \ne i}} u_j \right) \\ &= (-1)^n \sum_{i=0}^{n} \ell_i k_i u_i \left(\prod_{\substack{0 \le j \le n \\ j \ne i}} u_j \right) \\ &= (-1)^n \left(\sum_{i=0}^{n} \ell_i k_i \right) u_0 u_1 \cdots u_n, \end{aligned}$$

showing that ab is a multiple of $u_0 u_1 \cdots u_n$, as required. This completes the proof. □

Remark 3. Lemma 2 is inspired by Farhi's result [5, Theorem 2] which becomes a special case for $\mathcal{A} = \mathbb{Z}$, $a = \mathrm{lcm}(u_0, u_1, \dots, u_n)$, and $b = \mathrm{lcm}(\prod_{\substack{0 \le j \le n \\ j \ne i}} (u_i - u_j); i = 0, 1, \dots, n)$. It was precisely this special case which led the second author [5] to establish the first nontrivial lower bounds for the least common multiples of arithmetic progressions.

Now, we use Lemma 2 to establish a new proof of Theorem 1, which is purely algebraic.

A new proof of Theorem 1. Since $L_{c,m,n}$ is non-increasing relative to m, then it suffices to prove the result of the theorem for $m = \lceil \frac{n}{2} \rceil$, that is $L_{c,\lceil \frac{n}{2} \rceil,n} \ge 2^n$. For simplicity, put $m_0 = \lceil \frac{n}{2} \rceil$. So, we have to show that $L_{c,m_0,n} \ge 2^n$. For $n \in \{1, 2, \dots, 6\}$, this can be easily checked by hand (as was done by Oon). Suppose in what follows that $n \ge 7$. It is well-known and easily proved that for any integer $r \ge 7$, we have $\lceil \frac{r}{2} \rceil \binom{r}{\lceil \frac{r}{2} \rceil} \ge 2^r$. According to this inequality for $r = n$, it suffices to show that $L_{c,m_0,n} \ge m_0 \binom{n}{m_0}$. More generally, we shall show that:

$$L_{c,m',n} \ge m' \binom{n}{m'} \qquad (\forall m' \in \mathbb{N}^*,\ m' \le n). \tag{9}$$

Let $m' \in \mathbb{N}^*$ such that $m' \le n$. To prove (9), we apply Lemma 2 for $\mathcal{A} = \mathbb{Z}[\sqrt{-c}]$ by taking for the u_i's the elements $m' + \sqrt{-c}, m' + 1 + \sqrt{-c}, \dots, n + \sqrt{-c}$ of $\mathcal{A}$ and for a and b the integers $a = L_{c,m',n}$ and $b = (n - m')!$. For any $k \in \{m', m' + 1, \dots, n\}$, since $L_{c,m',n}$ is a multiple (in $\mathbb{Z}$, so also in $\mathcal{A} = \mathbb{Z}[\sqrt{-c}]$) of $(k^2 + c)$ and $k^2 + c = \left(k + \sqrt{-c}\right)\left(k - \sqrt{-c}\right)$ is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $k + \sqrt{-c}$, then $L_{c,m',n}$ is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $k + \sqrt{-c}$. This shows that the first condition of Lemma 2 is satisfied. On the other hand, we have for all $k \in \{m', m' + 1, \dots, n\}$:

$$\prod_{\substack{m' \le \ell \le n \\ \ell \ne k}} \left(\left(k + \sqrt{-c}\right) - \left(\ell + \sqrt{-c}\right) \right) = \prod_{\substack{m' \le \ell \le n \\ \ell \ne k}} (k - \ell) = (-1)^{n-k} (k - m')! (n - k)!,$$

which divides (in $\mathbb{Z}$, so also in $\mathbb{Z}[\sqrt{-c}]$) the integer $(n-m')!$ (since $\frac{(n-m')!}{(k-m')!(n-k)!} = \binom{n-m'}{k-m'} \in \mathbb{Z}$). This shows that the second condition of Lemma 2 is also satisfied. We thus deduce (by applying Lemma 2) that $L_{c,m',n}(n-m')!$ is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $\prod_{k=m'}^{n}\left(k+\sqrt{-c}\right)$. So, there exist $x, y \in \mathbb{Z}$ such that:

$$L_{c,m',n}(n-m')! = \left(x + y\sqrt{-c}\right)\prod_{k=m'}^{n}\left(k+\sqrt{-c}\right). \tag{10}$$

Therefore, by taking the absolute value in $\mathbb{C}$ on both sides, we get

$$L_{c,m',n}(n-m')! = \sqrt{x^2+cy^2}\prod_{k=m'}^{n}\sqrt{k^2+c}.$$

Next, since $x^2+cy^2 \in \mathbb{N}$ and $x^2+cy^2 \neq 0$ (because $x^2+cy^2 = 0 \Longrightarrow L_{c,m',n} = 0$, which is false) then $x^2+cy^2 \geq 1$. Hence

$$L_{c,m',n} = \frac{\sqrt{x^2+cy^2}\prod_{k=m'}^{n}\sqrt{k^2+c}}{(n-m')!} \geq \frac{\prod_{k=m'}^{n}\sqrt{k^2+c}}{(n-m')!} \geq \frac{\prod_{k=m'}^{n}k}{(n-m')!} = m'\binom{n}{m'},$$

as required. This completes the proof of the theorem. □

Naturally, we have the following question:

> *How could we improve the lower bound $L_{c,m,n} \geq \frac{\prod_{k=m}^{n}\sqrt{k^2+c}}{(n-m)!}$, obtained during the proof of Theorem 1 and initially established by Oon [12]?*

To simplify, suppose that $c = 1$ and let $m, n \in \mathbb{N}^*$ such that $m \leq n$. According to Formula (10), the positive integer $L_{1,m,n}(n-m)!$ is a multiple (in $\mathbb{Z}[i]$) of the Gauss integer $\prod_{k=m}^{n}(k+i)$. Next, by taking the conjugates (in $\mathbb{C}$) of both sides of (10), we obtain that $L_{1,m,n}(n-m)!$ is also a multiple (in $\mathbb{Z}[i]$) of the Gauss integer $\prod_{k=m}^{n}(k-i)$. It follows from those two facts that $L_{1,m,n}(n-m)!$ is a multiple (in $\mathbb{Z}[i]$) of[1]

$$\begin{aligned}\mathrm{lcm}_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right) &= \frac{\prod_{k=m}^{n}(k+i)\cdot\prod_{k=m}^{n}(k-i)}{\gcd_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right)} \\ &= \frac{\prod_{k=m}^{n}(k^2+1)}{\gcd_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right)}.\end{aligned}$$

[1]Let $\mathcal{A}$ be an integral domain and let $a, b \in \mathcal{A}$. Then an element d of $\mathcal{A}$ is called a greatest common divisor of a and b (and denoted by $\gcd_{\mathcal{A}}(a,b)$) if d divides both a and b and if any other $d' \in \mathcal{A}$, which divides both a and b, also divides d. Similarly, an element m of $\mathcal{A}$ is called a least common multiple of a and b (and denoted by $\mathrm{lcm}_{\mathcal{A}}(a,b)$) if m is a multiple of both a and b and if any other $m' \in \mathcal{A}$, which is a multiple of both a and b, is also a multiple of m. Note that $\gcd_{\mathcal{A}}(a,b)$ and $\mathrm{lcm}_{\mathcal{A}}(a,b)$ exist at least when $\mathcal{A}$ is a unique factorization domain (which is the case of $\mathbb{Z}[i]$) and they are unique up to a multiplication by a unit.

Consequently

$$L_{1,m,n} \geq \frac{\prod_{k=m}^{n}(k^2+1)}{(n-m)!\left|\gcd_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right)\right|}. \tag{11}$$

Remarkably, the trivial upper bound

$$\left|\gcd_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right)\right| \leq \left|\prod_{k=m}^{n}(k+i)\right| \leq \prod_{k=m}^{n}\sqrt{k^2+1}$$

suffices to establish the Oon lower bound $L_{1,m,n} \geq \frac{\prod_{k=m}^{n}\sqrt{k^2+1}}{(n-m)!}$. So, a nontrivial upper bound for the number $\left|\gcd_{\mathbb{Z}[i]}\left(\prod_{k=m}^{n}(k+i), \prod_{k=m}^{n}(k-i)\right)\right|$ certainly gives an improvement of the Oon theorem. On the other hand, for $a, b \in \mathbb{Z}$ such that $(a, b) \neq (0, 0)$, we can easily check that $\gcd_{\mathbb{Z}[i]}(a+bi, a-bi)$ is not far from $\gcd_{\mathbb{Z}}(a, b)$. Precisely, we have

$$\gcd_{\mathbb{Z}[i]}(a+bi, a-bi) = (\sigma + i\tau)\gcd_{\mathbb{Z}}(a, b),$$

where $\sigma, \tau \in \{-1, 0, 1\}$ and $(\sigma, \tau) \neq (0, 0)$. So, for the case $c = 1$, we are led to study the arithmetic function:

$$\begin{array}{rccc} h: & \mathbb{Z}[i] \setminus \{0\} & \longrightarrow & \mathbb{N}^* \\ & a+bi & \longmapsto & \gcd(a, b) \end{array},$$

and precisely to find nontrivial upper bounds for the quantities $h\left(\prod_{k=m}^{n}(k+i)\right)$ $(m, n \in \mathbb{N}^*,\ m \leq n)$. For the general case ($c \in \mathbb{N}^*$), the arithmetic function we need to study is given by:

$$\begin{array}{rccc} h_c: & \mathbb{Z}[\sqrt{-c}] \setminus \{0\} & \longrightarrow & \mathbb{N}^* \\ & a+b\sqrt{-c} & \longmapsto & \gcd(a, b) \end{array}$$

and the quantities we need to bound from above are $h_c\left(\prod_{k=m}^{n}(k+\sqrt{-c})\right)$ $(m, n \in \mathbb{N}^*,\ m \leq n)$.

The following proposition has as objective to replace a specific arithmetic language of the ring $\mathbb{Z}[\sqrt{-c}]$ by its simpler analog in $\mathbb{Z}$.

Proposition 4. *Let $c \in \mathbb{N}^*$ and $N, a, b \in \mathbb{Z}$, with $(a, b) \neq (0, 0)$. Then N is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $\left(a + b\sqrt{-c}\right)$ if and only if N is a multiple (in $\mathbb{Z}$) of $\frac{a^2+cb^2}{\gcd(a,b)}$.*

Proof. The result of the proposition is trivial for $b = 0$. Suppose in what follows that $b \neq 0$.

Suppose that N is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $\left(a + b\sqrt{-c}\right)$; that is there exist $x, y \in \mathbb{Z}$ such that:

$$N = \left(x + y\sqrt{-c}\right)\left(a + b\sqrt{-c}\right).$$

By identifying the real and imaginary parts of the two hand-sides of this equality, we get

$$N = ax - byc, \tag{12}$$

$$0 = bx + ay. \tag{13}$$

Next, putting $d := \gcd(a,b)$, there exist $a', b' \in \mathbb{Z}$, with $b' \neq 0$ and $\gcd(a', b') = 1$, such that $a = da'$ and $b = db'$. By substituting these in (13), we obtain (after simplifying)

$$b'x = -a'y. \tag{14}$$

This last equality shows that b' divides $a'y$. But since $\gcd(a', b') = 1$, then (according to the Gauss lemma) b' divides y. So there exists $k \in \mathbb{Z}$ such that $y = kb'$. By reporting this in (14), we get $x = -ka'$. Then by substituting $x = -ka' = -k\frac{a}{d}$ and $y = kb' = k\frac{b}{d}$ in (12), we finally obtain

$$N = -k\frac{a^2 + cb^2}{d} = -k\frac{a^2 + cb^2}{\gcd(a,b)},$$

showing that N is a multiple (in $\mathbb{Z}$) of $\frac{a^2+cb^2}{\gcd(a,b)}$, as required.

Conversely, suppose that N is a multiple (in $\mathbb{Z}$) of $\frac{a^2+cb^2}{\gcd(a,b)}$. Then there exists $k \in \mathbb{Z}$ such that:

$$N = k\frac{a^2 + cb^2}{\gcd(a,b)} = k\frac{a - b\sqrt{-c}}{\gcd(a,b)}\left(a + b\sqrt{-c}\right) = \left(k\frac{a}{\gcd(a,b)} - k\frac{b}{\gcd(a,b)}\sqrt{-c}\right)\left(a + b\sqrt{-c}\right).$$

Since $\left(k\frac{a}{\gcd(a,b)} - k\frac{b}{\gcd(a,b)}\sqrt{-c}\right) \in \mathbb{Z}[\sqrt{-c}]$, the last equality shows that N is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $\left(a + b\sqrt{-c}\right)$, as required. This completes the proof of the proposition. $\square$

From Proposition 4, we derive the following corollary, which is the first key step to obtaining the results of this paper.

Corollary 5. *Let $c, m, n \in \mathbb{N}^*$ such that $m \leq n$. Then the positive integer $L_{c,m,n}(n-m)!$ is a multiple (in $\mathbb{Z}$) of the positive integer:*

$$\frac{\prod_{k=m}^{n}\left(k^2 + c\right)}{h_c\left(\prod_{k=m}^{n}\left(k + \sqrt{-c}\right)\right)}.$$

Proof. Formula (10) (obtained during our new proof of Theorem 1) shows that $L_{c,m,n}(n-m)!$ is a multiple (in $\mathbb{Z}[\sqrt{-c}]$) of $\prod_{k=m}^{n}\left(k + \sqrt{-c}\right)$. But, according to Proposition 4, this last property is equivalent to the statement of the corollary. $\square$

In view of Corollary 5, to bound from below $L_{c,m,n}$ $(c, m, n \in \mathbb{N}^*,\ m \leq n)$, it suffices to bound from above $h_c\left(\prod_{k=m}^{n}(k + \sqrt{-c})\right)$. Likewise, to find a nontrivial (rational) divisor of $L_{c,m,n}$, it suffices to find a nontrivial multiple of $h_c\left(\prod_{k=m}^{n}(k + \sqrt{-c})\right)$. This is what we will do in what follows.

2.2 An explicit Bézout identity

In the following, let $c \in \mathbb{N}^*$ and $k \in \mathbb{N}$ be fixed and define

$$\begin{aligned} P_k(X) &:= \left(X+\sqrt{-c}\right)\left(X-1+\sqrt{-c}\right)\cdots\left(X-k+\sqrt{-c}\right) := A_k(X)+B_k(X)\sqrt{-c}, \\ \overline{P_k}(X) &:= \left(X-\sqrt{-c}\right)\left(X-1-\sqrt{-c}\right)\cdots\left(X-k-\sqrt{-c}\right) := A_k(X)-B_k(X)\sqrt{-c}, \end{aligned}$$

where we understand that $A_k, B_k \in \mathbb{Z}[X]$. In what follows, we find nontrivial multiples for the positive integers $h_c(P_k(n)) = \gcd(A_k(n), B_k(n))$ $(n \geq 1)$. To do so, we look for two polynomial sequences $(a_k(n))_n$ and $(b_k(n))_n$ so that the polynomial sequence $(a_k(n)A_k(n) + b_k(n)B_k(n))_n$ be independent on n. This leads to looking for two polynomials $U_k, V_k \in \mathbb{Q}[X]$ which satisfy the Bézout identity:

$$U_k(X)\,A_k(X) + V_k(X)\,B_k(X) = 1.$$

Next, since $A_k = \frac{P_k+\overline{P_k}}{2}$ and $B_k = \frac{P_k-\overline{P_k}}{2\sqrt{-c}}$, the problem is equivalent to looking for $\sigma_k, \tau_k \in \mathbb{Q}\left(\sqrt{-c}\right)[X]$ such that:

$$\sigma_k(X)\,P_k(X) + \tau_k(X)\,\overline{P_k}(X) = 1.$$

Let us first justify the existence of such σ_k and τ_k. Denoting by $Z(P)$ the set of all the complex zeros of a polynomial $P \in \mathbb{C}[X]$, we have

$$Z(P_k) = \left\{-\sqrt{-c}, 1-\sqrt{-c}, \dots, k-\sqrt{-c}\right\} \text{ and } Z\left(\overline{P_k}\right) = \left\{\sqrt{-c}, 1+\sqrt{-c}, \dots, k+\sqrt{-c}\right\},$$

showing that $Z(P_k) \cap Z\left(\overline{P_k}\right) = \emptyset$; that is P_k and $\overline{P_k}$ do not have a common zero in $\mathbb{C}$. This implies that P_k and $\overline{P_k}$ are coprime in $\mathbb{C}[X]$; so coprime also in $\mathbb{Q}\left(\sqrt{-c}\right)[X]$ (since $P_k, \overline{P_k} \in \mathbb{Q}\left(\sqrt{-c}\right)[X]$). It follows (according to Bézout's theorem) that there exist $\sigma_k, \tau_k \in \mathbb{Q}\left(\sqrt{-c}\right)[X]$ such that: $\sigma_k P_k + \tau_k \overline{P_k} = 1$, as required.

Now, to find explicitly such σ_k and τ_k, we need the following more precise version of Bézout's theorem:

Theorem 6. *Let $\mathbb{K}$ be a field and P and Q be two non-constant polynomials of $\mathbb{K}[X]$ such that $\gcd_{\mathbb{K}[X]}(P,Q) = 1$. Then there exists a unique pair (U,V) of polynomials of $\mathbb{K}[X]$, with $\deg U < \deg Q$ and $\deg V < \deg P$, such that:*

$$PU + QV = 1.$$

Proof. Since $\gcd_{\mathbb{K}[X]}(P,Q) = 1$, then (according to Bézout's theorem) there exist $U_0, V_0 \in \mathbb{K}[X]$ such that:

$$PU_0 + QV_0 = 1.$$

Next, consider in $\mathbb{K}[X]$ the euclidean division of U_0 by Q and the euclidean division of V_0 by $(-P)$:

$$\begin{aligned} U_0 &= U_1 Q + U, \\ V_0 &= V_1(-P) + V, \end{aligned}$$

where $U_1, V_1, U, V \in \mathbb{K}[X]$, $\deg U < \deg Q$, and $\deg V < \deg(-P) = \deg P$. So, we have

$$PU + QV = P(U_0 - U_1 Q) + Q(V_0 + V_1 P) = PQ(V_1 - U_1) + PU_0 + QV_0 = PQ(V_1 - U_1) + 1.$$

If $V_1 - U_1 \neq 0$, then the last equality implies that $\deg(PU + QV) \geq \deg(PQ)$, which is impossible, since $\deg U < \deg Q$ and $\deg V < \deg P$. Thus $V_1 - U_1 = 0$, which gives $PU + QV = 1$. The existence of the pair (U, V) as required by the theorem is proved. It remains to prove the uniqueness of (U, V). Let (U_*, V_*) another pair of polynomials of $\mathbb{K}[X]$, with $\deg U_* < \deg Q$, $\deg V_* < \deg P$, and $PU_* + QV_* = 1$, and let us prove that $(U_*, V_*) = (U, V)$. We have

$$P(UV_* - U_* V) = (PU)V_* - (PU_*)V = (1 - QV)V_* - (1 - QV_*)V = V_* - V,$$

showing that the polynomial $(V_* - V)$ is a multiple of P in $\mathbb{K}[X]$. But since $\deg(V_* - V) < \deg P$ (because $\deg V < \deg P$ and $\deg V_* < \deg P$), we have inevitably $V_* - V = 0$; hence $V_* = V$. Using this, we get $PU_* = 1 - QV_* = 1 - QV = PU$. Thus $U_* = U$. Consequently, we have $(U_*, V_*) = (U, V)$, as required. This completes the proof of the theorem. □

In our context, the application of Theorem 6 gives the following corollary:

Corollary 7. *There exists a unique polynomial $\sigma_k \in \mathbb{C}[X]$, with degree $\leq k$, such that:*

$$\sigma_k P_k + \overline{\sigma_k}\,\overline{P_k} = 1.$$

Proof. According to Theorem 6 (applied for $\mathbb{K} = \mathbb{C}$ and $(P, Q) = \left(P_k, \overline{P_k}\right)$), there exists a unique pair (σ_k, τ_k) of polynomials of $\mathbb{C}[X]$, with $\deg \sigma_k < \deg \overline{P_k} = k + 1$ and $\deg \tau_k < \deg P_k = k + 1$, such that $\sigma_k P_k + \tau_k \overline{P_k} = 1$. By taking the conjugates in $\mathbb{C}[X]$ of both sides of the last equality, we derive that $\overline{\sigma_k}\,\overline{P_k} + \overline{\tau_k} P_k = 1$, that is $\overline{\tau_k} P_k + \overline{\sigma_k}\,\overline{P_k} = 1$. Since $\deg \overline{\tau_k} = \deg \tau_k < k + 1$ and $\deg \overline{\sigma_k} = \deg \sigma_k < k + 1$, this shows that the pair $(\overline{\tau_k}, \overline{\sigma_k})$ satisfies the property that characterizes (σ_k, τ_k). Thus $(\overline{\tau_k}, \overline{\sigma_k}) = (\sigma_k, \tau_k)$, that is $\tau_k = \overline{\sigma_k}$. Consequently, we have $\sigma_k P_k + \overline{\sigma_k}\,\overline{P_k} = 1$. This completes the proof of the corollary. □

Now, we are going to determine the explicit expression of the polynomial σ_k announced by Corollary 7. By replacing, in the identity $\sigma_k(X) P_k(X) + \overline{\sigma_k}(X)\,\overline{P_k}(X) = 1$, the indeterminate X by the numbers $s + \sqrt{-c}$ $(s = 0, 1, \dots, k)$, we get

$$\sigma_k\left(s + \sqrt{-c}\right) = \frac{1}{P_k\left(s + \sqrt{-c}\right)} \quad (\forall s \in \{0, 1, \dots, k\}). \tag{15}$$

(since $\overline{P_k}\left(s + \sqrt{-c}\right) = 0$ for $s = 0, 1, \dots, k$). So the values of σ_k are known for $(k+1)$ equidistant points with distance 1. Since $\deg \sigma_k \leq k$, this is sufficient to determine the expression of $\sigma_k(X)$ by using for example the Newton forward interpolation formula. Doing so, we obtain that:

$$\sigma_k(X) = \sum_{\ell=0}^{k} \frac{\left(\Delta^\ell \sigma_k\right)\left(\sqrt{-c}\right)}{\ell!} \left(X - \sqrt{-c}\right)^{\underline{\ell}}.$$

Then by using (1), we derive that:

$$\begin{aligned}\sigma_k\left(X\right) &= \sum_{\ell=0}^{k}\sum_{j=0}^{\ell}\frac{(-1)^{\ell-j}}{\ell!}\binom{\ell}{j}\sigma_k\left(j+\sqrt{-c}\right)\left(X-\sqrt{-c}\right)^{\ell} \\ &= \sum_{\ell=0}^{k}\left(\frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\sigma_k\left(j+\sqrt{-c}\right)\right)\left(X-\sqrt{-c}\right)^{\ell} \\ &= \sum_{\ell=0}^{k}\left(\frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{P_k\left(j+\sqrt{-c}\right)}\right)\left(X-\sqrt{-c}\right)^{\ell}\end{aligned}$$

(according to (15)). So, by setting for all $\ell \in \{0,1,\dots,k\}$:

$$\Theta_{k,\ell} := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{P_k(j+\sqrt{-c})}, \tag{16}$$

we get

$$\sigma_k\left(X\right) = \sum_{\ell=0}^{k}\Theta_{k,\ell}\left(X-\sqrt{-c}\right)^{\ell}. \tag{17}$$

It remains to simplify the expressions of the numbers $\Theta_{k,\ell}$ $(0 \leq \ell \leq k)$. To do so, we introduce the functions $R_{k,\ell}$ $(0 \leq \ell \leq k)$, defined by:

$$R_{k,\ell}(z) := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{P_k(z+j+\sqrt{-c})}, \tag{18}$$

so that we have

$$\Theta_{k,\ell} = R_{k,\ell}(0) \qquad (\forall \ell \in \{0,1,\dots,k\}). \tag{19}$$

Since the $R_{k,\ell}$'s $(0 \leq \ell \leq k)$ are all rational functions, each of them is holomorphic on its domain of definition, that is each $R_{k,\ell}$ $(0 \leq \ell \leq k)$ is holomorphic on

$$D_\ell := \mathbb{C}\setminus\left\{j-2\sqrt{-c}\ ;\ j\in\mathbb{Z}\text{ and }-\ell\leq j\leq k\right\}.$$

Consequently, the common domain of holomorphy of the functions $R_{k,\ell}$ $(0 \leq \ell \leq k)$ is the open connected region D of $\mathbb{C}$, given by:

$$D := \bigcap_{0\leq\ell\leq k} D_\ell = \mathbb{C}\setminus\{j-2\sqrt{-c}\ ;\ j\in\mathbb{Z}\text{ and }-k\leq j\leq k\}.$$

Using the principle of analytical continuation together with the theory of the gamma and beta functions, we can find another expression of $R_{k,\ell}$ $(0 \leq \ell \leq k)$, which is simpler than the above. We have the following proposition:

Proposition 8. *For all $\ell \in \mathbb{N}$, with $\ell \leq k$, and all $z \in D$, we have*

$$R_{k,\ell}(z) = \frac{(-1)^{k+\ell}}{z+2\sqrt{-c}}\binom{k+\ell}{\ell}\frac{1}{\left(k-2\sqrt{-c}-z\right)^{\underline{k}}\left(\ell+2\sqrt{-c}+z\right)^{\underline{\ell}}}. \tag{20}$$

Proof. Let $\ell \in \mathbb{N}$ such that $\ell \leq k$. According to the principle of analytical continuation, it suffices to prove Formula (20) for $z \in \mathbb{C}$, such that $\Re(z) > k$. For a such z, we have

$$\begin{aligned}
R_{k,\ell}(z) &:= \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{P_k(z+j+\sqrt{-c})} \\
&= \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{\left(z+j+2\sqrt{-c}\right)\left(z+j-1+2\sqrt{-c}\right)\cdots\left(z+j-k+2\sqrt{-c}\right)} \\
&= \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{\Gamma\left(z+j-k+2\sqrt{-c}\right)}{\Gamma\left(z+j+1+2\sqrt{-c}\right)} \\
&= \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\frac{1}{k!}\beta\left(z+j-k+2\sqrt{-c},k+1\right) \\
&= \frac{1}{k!\ell!}\sum_{j=0}^{\ell}\left((-1)^{\ell-j}\binom{\ell}{j}\int_0^1 t^{z+j-k-1+2\sqrt{-c}}(1-t)^k\mathrm{d}t\right) \\
&= \frac{1}{k!\ell!}\int_0^1 t^{z-k-1+2\sqrt{-c}}(1-t)^k\left(\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}t^j\right)\mathrm{d}t \\
&= \frac{1}{k!\ell!}\int_0^1 t^{z-k-1+2\sqrt{-c}}(1-t)^k\,(t-1)^\ell\,\mathrm{d}t \\
&= \frac{(-1)^\ell}{k!\ell!}\int_0^1 t^{z-k-1+2\sqrt{-c}}(1-t)^{k+\ell}\mathrm{d}t \\
&= \frac{(-1)^\ell}{k!\ell!}\beta\left(z-k+2\sqrt{-c},k+\ell+1\right) \\
&= \frac{(-1)^\ell}{k!\ell!}\frac{\Gamma\left(z-k+2\sqrt{-c}\right)\Gamma\left(k+\ell+1\right)}{\Gamma\left(z+\ell+1+2\sqrt{-c}\right)} \\
&= (-1)^\ell\binom{k+\ell}{\ell}\frac{1}{\left(z+\ell+2\sqrt{-c}\right)\left(z+\ell-1+2\sqrt{-c}\right)\cdots\left(z-k+2\sqrt{-c}\right)} \\
&= \frac{(-1)^{k+\ell}}{z+2\sqrt{-c}}\binom{k+\ell}{\ell}\frac{1}{\left(k-2\sqrt{-c}-z\right)^{\underline{k}}\left(\ell+2\sqrt{-c}+z\right)^{\underline{\ell}}},
\end{aligned}$$

as required. This completes the proof. □

From Proposition 8, we immediately derive a simpler explicit expression of $\sigma_k\left(X\right)$. We have the following corollary:

Corollary 9. *We have*

$$\sigma_k\left(X\right) = \frac{1}{2\sqrt{-c}\left(k-2\sqrt{-c}\right)^{\underline{k}}}\sum_{\ell=0}^{k}\frac{(-1)^{k+\ell}\binom{k+\ell}{\ell}}{\left(\ell+2\sqrt{-c}\right)^{\underline{\ell}}}\left(X-\sqrt{-c}\right)^{\underline{\ell}}.$$

Proof. This immediately follows from Formulas (17), (19), and (20). □

2.3 Nontrivial multiples of some values of h_c

In this subsection, we preserve the notation of Subsection 2.2. From Corollary 9, we derive the following theorem:

Theorem 10. *For all $c, n, m \in \mathbb{N}^*$, with $m \leq n$, we have*

$$h_c\left(\prod_{\ell=m}^{n}\left(\ell+\sqrt{-c}\right)\right) \quad \text{divides} \quad c\prod_{\ell=1}^{n-m}(\ell^2+4c).$$

Proof. Let $c, n, m \in \mathbb{N}^*$, with $m \leq n$. Putting $k := n-m \in \mathbb{N}$ and $d := c\prod_{\ell=1}^{n-m}(\ell^2+4c) \in \mathbb{N}^*$, we have $\prod_{\ell=m}^{n}\left(\ell+\sqrt{-c}\right) = P_k(n)$; so, we have to show that $h_c\left(P_k(n)\right)$ divides d. By noting that $2d = \sqrt{-c}\cdot 2\sqrt{-c}\left(k-2\sqrt{-c}\right)^{\underline{k}}\left(k+2\sqrt{-c}\right)^{\underline{k}}$, we derive from Corollary 9 that $2d\sigma_k \in \mathbb{Z}[\sqrt{-c}][X]$. So, there exist $r_k, s_k \in \mathbb{Z}[X]$ such that:

$$2d\sigma_k\left(X\right) = r_k\left(X\right) + s_k\left(X\right)\sqrt{-c}.$$

Next, the identity of polynomials $\sigma_k P_k + \overline{\sigma_k}\,\overline{P_k} = 1$ (given by Corollary 7) implies that $2d\sigma_k \cdot P_k + \overline{2d\sigma_k}\cdot\overline{P_k} = 2d$. By substituting in this last equality P_k by $\left(A_k + B_k\sqrt{-c}\right)$ and $2d\sigma_k$ by $\left(r_k + s_k\sqrt{-c}\right)$, we obtain (in particular) that:

$$r_k A_k - c s_k B_k = d,$$

implying that $\gcd_{\mathbb{Z}[X]}\left(A_k, B_k\right)$ divides d. We then conclude that $\gcd_{\mathbb{Z}}\left(A_k(n), B_k(n)\right) = h_c\left(P_k(n)\right)$ divides d, as required. □

2.4 New estimates for the number $L_{c,m,n}$

We have the following theorem:

Theorem 11. *Let $c, m, n \in \mathbb{N}^*$ such that $m \leq n$. Then:*

1. *The positive integer $L_{c,m,n}$ is a multiple of the rational number*

$$\frac{\prod_{k=m}^{n}\left(k^2+c\right)}{c\cdot(n-m)!\prod_{k=1}^{n-m}\left(k^2+4c\right)}.$$

2. *We have*

$$L_{c,m,n} \geq \lambda_1(c) \cdot m^2 \frac{n!^2}{m!^2(n-m)!^3},$$

where $\lambda_1(c) := e^{-\frac{2\pi^2}{3}c}/c$.

Proof. The first point of the theorem is an immediate consequence of Corollary 5 and Theorem 10. Next, using the well-known inequality $1 + x \leq e^x$ ($\forall x \in \mathbb{R}$), we have

$$\begin{aligned}
\prod_{k=1}^{n-m}\left(k^2+4c\right) &= \prod_{k=1}^{n-m} k^2\left(1+\frac{4c}{k^2}\right) \\
&= (n-m)!^2 \prod_{k=1}^{n-m}\left(1+\frac{4c}{k^2}\right) \\
&\leq (n-m)!^2 \prod_{k=1}^{n-m} e^{\frac{4c}{k^2}} \\
&\leq (n-m)!^2 \prod_{k=1}^{+\infty} e^{\frac{4c}{k^2}} \\
&= (n-m)!^2 e^{4c\sum_{k=1}^{+\infty}\frac{1}{k^2}} \\
&= (n-m)!^2 e^{4c(\frac{\pi^2}{6})} \\
&= (n-m)!^2 e^{\frac{2\pi^2}{3}c}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\prod_{k=m}^{n}\left(k^2+c\right)}{c\cdot(n-m)!\prod_{k=1}^{n-m}\left(k^2+4c\right)} &\geq \frac{\prod_{k=m}^{n} k^2}{c\cdot(n-m)!\cdot(n-m)!^2 e^{\frac{2\pi^2}{3}c}} \\
&= \frac{m^2\left(\frac{n!}{m!}\right)^2}{c\cdot(n-m)!^3 e^{\frac{2\pi^2}{3}c}} \\
&= \frac{e^{-\frac{2\pi^2}{3}c}}{c}\cdot m^2\frac{n!^2}{m!^2(n-m)!^3}.
\end{aligned}$$

The second point of the theorem then follows from the first one. This completes the proof. □

We shall now impose conditions on m (in terms of n) in order to optimize (resp., simplify) the estimate of the second point of Theorem 11. To do so, we first need to get rid of the factorials in that estimate. We have the following:

Corollary 12. *Let $c, n, m \in \mathbb{N}^*$ such that $m < n$. Then we have*

$$L_{c,m,n} \geq \lambda_2(c) \cdot \frac{nm}{(n-m)^{3/2}} \left(\frac{m^2}{(n-m)^3} \right)^{n-m} e^{3(n-m)}, \tag{21}$$

where $\lambda_2(c) := \frac{e^{-\frac{2\pi^2}{3}c - \frac{5}{12}}}{(2\pi)^{3/2}c}$.

Proof. Starting from the lower bound established by Theorem 11 for $L_{c,m,n}$ and estimating each of its factorial terms by using the well-known double inequality:

$$k^k e^{-k} \sqrt{2\pi k} \leq k! \leq k^k e^{-k} \sqrt{2\pi k} e^{\frac{1}{12k}} \qquad (\forall k \in \mathbb{N}^*)$$

(see, e.g., [4, Problem 1.15]), we get

$$L_{c,m,n} \geq \lambda_1(c)(2\pi)^{-3/2} \cdot \frac{nm}{(n-m)^{3/2}} \cdot \left(\frac{n}{m}\right)^{2n} \cdot \left(\frac{m^2}{(n-m)^3} \right)^{n-m} e^{n-m} \cdot e^{-\frac{1}{6m} - \frac{1}{4(n-m)}}.$$

Next, since $e^{-\frac{1}{6m} - \frac{1}{4(n-m)}} \geq e^{-\frac{1}{6} - \frac{1}{4}} = e^{-\frac{5}{12}}$ and $\left(\frac{n}{m}\right)^{2n} = e^{-2n\log(\frac{m}{n})} \geq e^{-2n(\frac{m}{n}-1)} = e^{2(n-m)}$, then we deduce that:

$$L_{c,m,n} \geq \lambda_1(c)(2\pi)^{-3/2} e^{-5/12} \cdot \frac{nm}{(n-m)^{3/2}} \left(\frac{m^2}{(n-m)^3} \right)^{n-m} e^{3(n-m)},$$

as required. □

In the context of Corollary 12, by supposing that $n - m$ is of order of magnitude n^α for large n (where $0 < \alpha < 1$), then the dominant part of the lower bound (21) for $L_{c,m,n}$ is $\left(\frac{m^2}{(n-m)^3}\right)^{n-m}$ and has order of magnitude $n^{(2-3\alpha)n^\alpha}$. So, to have an optimal estimate, we must take α less than but not too far from $\frac{2}{3}$ (a study of the function $\alpha \longmapsto (2-3\alpha)n^\alpha$ shows that the best value of α is $\alpha = \frac{2}{3} - \frac{1}{\log n}$). A concrete result specifying this heuristic reasoning is given by the following theorem:

Theorem 13. *Let $c, m, n \in \mathbb{N}^*$ such that $m \leq n - \frac{1}{2}n^{2/3}$. Then we have*

$$L_{c,m,n} \geq \lambda_3(c) \cdot \left(n - \frac{1}{2}n^{2/3}\right) \cdot \left(2e^3\right)^{\left\lfloor \frac{1}{2}n^{2/3} \right\rfloor},$$

where $\lambda_3(c) := \frac{e^{-\frac{2\pi^2}{3}c - \frac{5}{12}}}{\pi^{3/2}c}$.

Proof. A simple calculation shows that the result of the theorem is true for $n < 3$. Suppose in what follows that $n \geq 3$ and let $m_n := n - \left\lfloor \frac{1}{2}n^{2/3} \right\rfloor < n$; so $m \leq m_n$. From Corollary 12, we have

$$\begin{aligned}
L_{c,m_n,n} &\geq \lambda_2(c)\frac{n\left(n-\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor\right)}{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor^{3/2}}\left(\frac{\left(n-\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor\right)^2}{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor^3}\right)^{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor} e^{3\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor} \\
&\geq \lambda_2(c)\frac{n\left(n-\frac{1}{2}n^{2/3}\right)}{\left(\frac{1}{2}n^{2/3}\right)^{3/2}}\left(\frac{\left(n-\frac{1}{2}n^{2/3}\right)^2}{\left(\frac{1}{2}n^{2/3}\right)^3}\right)^{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor} e^{3\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor} \\
&= 2^{3/2}\lambda_2(c)\left(n-\frac{1}{2}n^{2/3}\right)\left(8\left(1-\frac{1}{2n^{1/3}}\right)^2\right)^{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor} e^{3\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor}.
\end{aligned}$$

But since $1-\frac{1}{2n^{1/3}} \geq \frac{1}{2}$ (because $n \geq 1$), we deduce that:

$$L_{c,m_n,n} \geq 2^{3/2}\lambda_2(c)\left(n-\frac{1}{2}n^{2/3}\right)\left(2e^3\right)^{\left\lfloor\frac{1}{2}n^{2/3}\right\rfloor}.$$

The required result follows from the trivial fact that $L_{c,m,n} \geq L_{c,m_n,n}$ (since $m \leq m_n$). □

In another direction, we derive from Corollary 12 the following theorem, which completes (in a way) Theorem 13 above.

Theorem 14. *Let $c, m, n \in \mathbb{N}^*$ such that $n - \frac{1}{2}n^{2/3} \leq m \leq n$. Then we have*

$$L_{c,m,n} \geq \lambda_2(c) \cdot ne^{3(n-m)},$$

where $\lambda_2(c)$ is defined in Corollary 12.

Proof. The result of the theorem is trivial for $m = n$. Suppose in what follows that $m < n$; so we have $n \geq 2$. Now, let $f : [0, n] \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 - (n-x)^3$ ($\forall x \in [0, n]$). Since $f'(x) = 2x + 3(n-x)^2 > 0$ ($\forall x \in [0, n]$), then f is increasing. Next, we have

$$\begin{aligned}
f\left(n-\frac{1}{2}n^{2/3}\right) &= \left(n-\frac{1}{2}n^{2/3}\right)^2 - \left(\frac{1}{2}n^{2/3}\right)^3 \\
&= n^2 - n^{5/3} + \frac{1}{4}n^{4/3} - \frac{1}{8}n^2 \\
&= \frac{7}{8}n^2 - n^{5/3} + \frac{1}{4}n^{4/3}.
\end{aligned}$$

But since $n^2 \geq \frac{8}{7}n^{5/3}$ (because $n \geq 2$), it follows that $f\left(n-\frac{1}{2}n^{2/3}\right) \geq \frac{1}{4}n^{4/3} > 0$. So, the increase of f ensures that $f(m) > 0$ (since $m \geq n - \frac{1}{2}n^{2/3}$ by hypothesis). Thus $\frac{m^2}{(n-m)^3} > 1$ and $\frac{m}{(n-m)^{3/2}} > 1$. By reporting these into (21), we then conclude that:

$$L_{c,m,n} \geq \lambda_2(c) \cdot ne^{3(n-m)},$$

as required. This completes the proof. □

2.5 Comparison with Oon's lower bound

In the application of Oon's lower bound (i.e., Theorem 1), the number of the terms included in the least common multiple $\mathrm{lcm}(m^2 + c, \ldots, n^2 + c) = L_{c,m,n}$ must be $\geq n - \lceil \frac{n}{2} \rceil + 1 > \frac{n}{2}$; whereas when we put our theorems 13 and 14 together, this constraint is deleted. However, if the condition of application of Oon's theorem holds then we obtain a lower bound for $L_{c,m,n}$ stronger than those of our theorems. Further, our key result is rather the point 1 of Theorem 11 which provides a nontrivial rational divisor of $L_{c,m,n}$. Here, we have exploited this key result in a naive way. It is likely that "a smarter way" will provide better results.

3 Acknowledgments

The authors would like to thank the anonymous reviewer and the editor for their helpful comments and suggestions.

References

[1] P. Bateman, J. Kalb, and A. Stenger, A limit involving least common multiples: 10797, *Amer. Math. Monthly* **109** (2002), 393–394.

[2] P. L. Chebyshev, Mémoire sur les nombres premiers, *J. Math. Pures Appl.* **17** (1852), 366–390.

[3] J. Cilleruelo, The least common multiple of a quadratic sequence, *Compositio Math.* **147** (2011), 1129–1150.

[4] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers,* Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, 2012.

[5] B. Farhi, Nontrivial lower bounds for the least common multiple of some finite sequences of integers, *J. Number Theory* **125** (2007), 393–411.

[6] D. Hanson, On the product of the primes, *Canad. Math. Bull.* **15** (1972), 33–37.

[7] S. Hong and W. Feng, Lower bounds for the least common multiple of finite arithmetic progressions, *C. R. Acad. Sci. Paris* **I 343** (2006), 695–698.

[8] S. Hong and S. D. Kominers, Further improvements of lower bounds for the least common multiples of arithmetic progressions, *Proc. Amer. Math. Soc.* **138** (2010), 809–813.

[9] S. Hong, Y. Luo, G. Qian, and C. Wang, Uniform lower bound for the least common multiple of a polynomial sequence, *C. R. Acad. Sci. Paris* **351** (2013), 781–785.

[10] D. Kane and S. Kominers, Asymptotic improvements of lower bounds for the least common multiples of arithmetic progressions, *Canad. Math. Bull.* **57** (2014), 551–561.

[11] M. Nair, On Chebyshev-type inequalities for primes, *Amer. Math. Monthly* **89** (1982), 126–129.

[12] S. M. Oon, Note on the lower bound of least common multiple, *Abstr. Appl. Anal.* (2013), Article ID 218125.

2010 *Mathematics Subject Classification*: Primary 11A05; Secondary 11B83, 13G05, 30B40, 33B15.
Keywords: least common multiple, quadratic sequence.

Received January 30 2020; revised versions received May 8 2020; May 14 2020. Published in *Journal of Integer Sequences*, June 11 2020.

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