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ON THE POSSIBLE QUANTITIES OF FIBONACCI NUMBERS THAT OCCUR IN SOME TYPES OF INTERVALS

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Abstract. In this paper, we show that for any integer $a \geq 2$, each of the intervals $[a^k, a^{k+1})$ ($k \in \mathbb{N}$) contains either $\left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$ or $\left\lceil \frac{\log a}{\log \Phi} \right\rceil$ Fibonacci numbers. In addition, the density (in $\mathbb{N}$) of the set of all natural numbers k for which the interval $[a^k, a^{k+1})$ contains exactly $\left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$ Fibonacci numbers is equal to $\left(1 - \left\langle \frac{\log a}{\log \Phi} \right\rangle\right)$ and the density of the set of all natural numbers k for which the interval $[a^k, a^{k+1})$ contains exactly $\left\lceil \frac{\log a}{\log \Phi} \right\rceil$ Fibonacci numbers is equal to $\left\langle \frac{\log a}{\log \Phi} \right\rangle$.

1. Introduction and the main result

Throughout this paper, if x is a real number, we let $\lfloor x \rfloor$, $\lceil x \rceil$, and $\langle x \rangle$, respectively denote the greatest integer $\leq x$, the least integer $\geq x$, and the fractional part of x. Furthermore, we let $\operatorname{Card} X$ denote the cardinal of a given finite set X. Finally, for any subset A of $\mathbb{N}$, we define the *density* of A as the following limit (if it exists):

$$\mathbf{d}(A) := \lim_{N \to +\infty} \frac{\operatorname{Card}(A \cap [1, N])}{N}.$$

It is clear that if $\mathbf{d}(A)$ exists, then $\mathbf{d}(A) \in [0, 1]$.

The Fibonacci sequence $(F_n)_{n \in \mathbb{N}}$ is defined by $F_0 = 0$, $F_1 = 1$, and for all $n \in \mathbb{N}$

$$F_{n+2} = F_n + F_{n+1} \tag{1.1}$$

A Fibonacci number is simply a term of the Fibonacci sequence. In this paper, we denote by $\mathscr{F}$ the set of the all Fibonacci numbers, that is,

$$\mathscr{F} := \{F_n \ , \ n \in \mathbb{N}\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}.$$

First, let us recall some important identities that will be useful in our proofs in Section 2. The Fibonacci sequence can be extended to the negative index n by rewriting the recurrence relation (1.1) as $F_n = F_{n+2} - F_{n+1}$. By induction, we easily show that for all $n \in \mathbb{Z}$, we have

$$F_{-n} = (-1)^{n+1} F_n \tag{1.2}$$

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(see [**2**, Chapter 5] for the details). A closed formula of F_n ($n \in \mathbb{Z}$) in terms of n is known and given by

$$F_n = \frac{1}{\sqrt{5}} \left(\Phi^n - \overline{\Phi}^n \right), \tag{1.3}$$

where $\Phi := \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\overline{\Phi} := \frac{1-\sqrt{5}}{2} = -\frac{1}{\Phi}$. Formula (1.3) is called "the Binet Formula" and there are many ways to prove it (see, e.g., [**1**, Chapter 8] or [**2**, Chapter 5]). Note that the real numbers Φ and $\overline{\Phi}$ are the roots of the quadratic equation

$$x^2 = x + 1.$$

More generally, we can show by induction (see, e.g., [**1**, Chapter 8]) that for $x \in \{\Phi, \overline{\Phi}\}$ and for all $n \in \mathbb{Z}$, we have

$$x^n = F_n x + F_{n-1}. \tag{1.4}$$

As remarked by Honsberger in [**1**, Chapter 8], Binet's formula (1.3) immediately follows from the last formula (1.4). On the other hand, the Fibonacci sequence satisfies the following important formula

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m \qquad (\text{for all } n, m \in \mathbb{Z}), \tag{1.5}$$

which we call "the addition formula". A nice and easily proof of (1.5) uses the formula (1.4). We can also prove (1.5) by using matrix calculations as in [**1**, Chapter 8].

As usual, we associate the Lucas sequence $(L_n)_{n\in\mathbb{Z}}$, defined by: $L_0 = 2$, $L_1 = 1$ and for all $n \in \mathbb{Z}$,

$$L_{n+2} = L_n + L_{n+1} \tag{1.6}$$

with the Fibonacci sequence $(F_n)_{n\in\mathbb{Z}}$. There are many connections and likenesses between the Fibonacci sequence and the Lucas sequence. For example, we have two following formulas (see [**1**, Chapter 8] or [**2**, Chapter 5]):

$$L_n = F_{n-1} + F_{n+1}, \tag{1.7}$$

$$L_n = \Phi^n + \overline{\Phi}^n, \tag{1.8}$$

which hold for any $n \in \mathbb{Z}$. For many other connections between the Fibonacci and the Lucas numbers, the reader can consult two references cited just above.

Fibonacci's sequence plays a very important role in theoretical and applied mathematics. During the last two centuries, arithmetic, algebraic, and analytic properties of the Fibonacci sequence have been investigated by several authors. One of those properties concerns the occurrence of the Fibonacci numbers in some types of intervals. For example, the French mathematician Gabriel Lamé (1795–1870) proved that there must be either four or five Fibonacci numbers with the same number of digits (see [**3**, page 29]). A generalization of this result consists in determining the possible quantities of Fibonacci numbers that belong to an interval of the form $[a^k, a^{k+1})$, where a and k are positive integers. In this direction, Honsberger [**1**, Chapter 8] proved the following theorem.

Theorem (Honsberger [**1**]). *Let a and k be any two positive integers. Then between the consecutive powers a^k and a^{k+1} there can never occur more than a Fibonacci numbers.*

However, Honsberger's theorem gives only an upper bound for the quantity of the Fibonacci numbers in question. Furthermore, it is not optimal because for $a = 10$, it gives a result that is weaker than Lamé's one. In this paper, we obtain the optimal generalization of Lamé's result with precisions concerning some densities. Our main result is the following theorem.

Theorem 1.1. *Let $a \geq 2$ be an integer. Then, any interval of the form $[a^k, a^{k+1})$ $(k \in \mathbb{N})$ contains either $\left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$ or $\left\lceil \frac{\log a}{\log \Phi} \right\rceil$ Fibonacci numbers.*
In addition, the density (in $\mathbb{N}$) of the set of all natural numbers k for which the interval $[a^k, a^{k+1})$ contains exactly $\left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$ Fibonacci numbers is equal to $\left(1 - \left\langle \frac{\log a}{\log \Phi} \right\rangle\right)$ and the density of the set of all natural numbers k for which the interval $[a^k, a^{k+1})$ contains exactly $\left\lceil \frac{\log a}{\log \Phi} \right\rceil$ Fibonacci numbers is equal to $\left\langle \frac{\log a}{\log \Phi} \right\rangle$.

2. The proof of the main result

The proof of our main result needs the following lemmas.

Lemma 2.1. *For all positive integers n, we have*

$$\Phi^{n-2} \leq F_n \leq \Phi^{n-1}.$$

In addition, the left-hand side of this double inequality is strict whenever $n \geq 3$ and its right-hand side is strict whenever $n \geq 2$.

Proof. We argue by induction on n. The double inequality of the lemma is clearly true for $n = 1$ and for $n = 2$. For a given integer $n \geq 3$, suppose that the double inequality of the lemma holds for any positive integers $m < n$. So it holds in particular, for $m = n - 1$ and for $m = n - 2$, that is

$$\Phi^{n-3} \leq F_{n-1} \leq \Phi^{n-2} \qquad \text{and} \qquad \Phi^{n-4} \leq F_{n-2} \leq \Phi^{n-3}.$$

By adding corresponding sides of the two last double inequalities and by taking account that $\Phi^{n-3} + \Phi^{n-4} = \Phi^{n-4}(\Phi + 1) = \Phi^{n-4}\Phi^2 = \Phi^{n-2}$ (since $\Phi + 1 = \Phi^2$), $\Phi^{n-2} + \Phi^{n-3} = \Phi^{n-1}$ (for the same reason), and $F_{n-1} + F_{n-2} = F_n$, we get

$$\Phi^{n-2} \leq F_n \leq \Phi^{n-1},$$

which is the double inequality of the lemma for the integer n. This achieves the induction and confirms the validity of the double inequality of the lemma for any positive integer n. We can show the second part of the lemma by the same way. □

Lemma 2.2. *For any integer $a \geq 2$, the real number $\frac{\log a}{\log \Phi}$ is irrational.*

Proof. Let $a \geq 2$ be an integer. We argue by contradiction. Suppose that $\frac{\log a}{\log \Phi}$ is rational. Since $\frac{\log a}{\log \Phi} > 0$, we can write $\frac{\log a}{\log \Phi} = \frac{r}{s}$, where r and s are positive

integers. This gives $\Phi^r = a^s$ and shows that $\Phi^r \in \mathbb{Z}$. Then, since $\overline{\Phi}^r = L_r - \Phi^r$ (according to (1.8)), it follows that $\overline{\Phi}^r \in \mathbb{Z}$. But on the other hand, we have $\left|\overline{\Phi}^r\right| = \left|\overline{\Phi}\right|^r \in (0,1)$ (since $\left|\overline{\Phi}\right| \in (0,1)$). We thus have a contradiction which confirms that the real number $\frac{\log a}{\log \Phi}$ is irrational. □

Lemma 2.3. *When the positive real number x tends to infinity, then we have*

$$\mathrm{Card}\left(\mathscr{F} \cap [1,x)\right) \ \sim \ \frac{\log x}{\log \Phi}.$$

Proof. For a given $x > 1$, let $\mathscr{F} \cap [1,x) = \{F_2, F_3, \ldots, F_{h+1}\}$ for a positive integer h. So we have $\mathrm{Card}(\mathscr{F} \cap [1,x)) = h$ and

$$F_{h+1} < x \leq F_{h+2},$$

which gives

$$\frac{\log F_{h+1}}{h \log \Phi} < \frac{\log x}{h \log \Phi} \leq \frac{\log F_{h+2}}{h \log \Phi}.$$

But because we have (according to the Binet formula (1.3))

$$\lim_{h \to +\infty} \frac{\log F_{h+1}}{h \log \Phi} = \lim_{h \to +\infty} \frac{\log F_{h+2}}{h \log \Phi} = 1,$$

it follows that $\lim_{h \to +\infty} \frac{\log x}{h \log \Phi} = 1$. Hence $h \sim_{+\infty} \frac{\log x}{\log \Phi}$, as required. □

Lemma 2.4. *For any $n \in \mathbb{Z}$, we have*

$$F_{2n-1} \geq F_n^2.$$

Proof. Let $n \in \mathbb{Z}$. According to the addition formula (1.5), we have

$$F_{2n-1} = F_{n+(n-1)} = F_n^2 + F_{n-1}^2 \geq F_n^2.$$

The lemma is proved. □

Lemma 2.5. *For all $n, m \in \mathbb{Z}$, we have*

$$F_{n+m} = F_n L_m + (-1)^{m+1} F_{n-m}.$$

Proof. Let $n, m \in \mathbb{Z}$. According to the addition formula (1.5), we have

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m \tag{2.1}$$

and

$$F_{n-m} = F_n F_{-m+1} + F_{n-1} F_{-m} = (-1)^m F_n F_{m-1} + (-1)^{m+1} F_{n-1} F_m$$

(according to (1.2)). Hence

$$(-1)^m F_{n-m} = F_n F_{m-1} - F_{n-1} F_m. \tag{2.2}$$

By adding corresponding sides of (2.1) and (2.2), we get

$$F_{n+m} + (-1)^m F_{n-m} = F_n \left(F_{m+1} + F_{m-1}\right) = F_n L_m$$

(according to (1.7)). Hence

$$F_{n+m} = F_n L_m + (-1)^{m+1} F_{n-m},$$

as required. □

Lemma 2.6 (the key lemma)**.** *For all $n, m \in \mathbb{N}$, satisfying $(n, m) \neq (0, 1)$ and $n \geq m - 1$, we have*

$$F_n \lfloor \Phi^m \rfloor \leq F_{n+m} \leq F_n \lceil \Phi^m \rceil .$$

Proof. The double inequality of the lemma is trivial for $m = 0$. For what follows, assume that $m \geq 1$. We distinguish two cases according to the parity of m.

1st case: (if m is even). In this case, we have $\overline{\Phi}^m \in (0, 1)$ (since $\overline{\Phi} \in (-1, 0)$ and m is even). Using (1.8), it follows that

$$\Phi^m = L_m - \overline{\Phi}^m \in (L_m - 1, L_m).$$

Consequently, we have

$$\lfloor \Phi^m \rfloor = L_m - 1 \qquad \text{and} \qquad \lceil \Phi^m \rceil = L_m.$$

So, for this case, we have to show that

$$F_n (L_m - 1) \leq F_{n+m} \leq F_n L_m.$$

Let us show the last double inequality. According to Lemma 2.5, we have

$$\begin{aligned} F_{n+m} &= F_n L_m + (-1)^{m+1} F_{n-m} \\ &= F_n L_m - F_{n-m} \qquad \text{(because } m \text{ is even)} \\ &= F_n (L_m - 1) + (F_n - F_{n-m}) . \end{aligned} \tag{2.3}$$

Next, since $n - m \geq -1$ (because $n \geq m - 1$ by hypothesis), then we have

$$F_{n-m} \geq 0,$$

and since $n \geq n - m \geq -1$ and $(n, n - m) \neq (0, -1)$ (because $(n, m) \neq (0, 1)$ by hypothesis), then we have $F_n \geq F_{n-m}$, that is,

$$F_n - F_{n-m} \geq 0.$$

Therefore, the second and the third equalities of (2.3) show that

$$F_n (L_m - 1) \leq F_{n+m} \leq F_n L_m,$$

as required.

2nd case: (if m is odd). In this case, we have $\overline{\Phi}^m \in (-1, 0)$ (because $\overline{\Phi} \in (-1, 0)$ and m is odd). Using (1.8), it follows that

$$\Phi^m = L_m - \overline{\Phi}^m \in (L_m, L_m + 1).$$

Consequently, we have

$$\lfloor \Phi^m \rfloor = L_m \qquad \text{and} \qquad \lceil \Phi^m \rceil = L_m + 1.$$

So, for this case, we have to show that

$$F_n L_m \leq F_{n+m} \leq F_n (L_m + 1) .$$

Let us show this last double inequality. According to Lemma 2.5, we have

$$\begin{aligned} F_{n+m} &= F_n L_m + (-1)^{m+1} F_{n-m} \\ &= F_n L_m + F_{n-m} \qquad \text{(because } m \text{ is odd)} \\ &= F_n \left(L_m + 1\right) - \left(F_n - F_{n-m}\right). \end{aligned} \tag{2.4}$$

For the same reasons as in the first case, we have

$$F_{n-m} \geq 0 \qquad \text{and} \qquad F_n - F_{n-m} \geq 0.$$

According to the second and the third equalities of (2.4), it follows that

$$F_n L_m \leq F_{n+m} \leq F_n \left(L_m + 1\right),$$

as required. This completes the proof of the lemma. □

We are now ready to prove our main result.

Proof of Theorem 1.1. Let $a \geq 2$ be a fixed integer. For simplification, we put for any natural number k, $I_k := [a^k, a^{k+1})$ and $\ell := \left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$. Since the real number $\frac{\log a}{\log \Phi}$ is not an integer (according to Lemma 2.2), we deduce that $\ell + 1 = \left\lceil \frac{\log a}{\log \Phi} \right\rceil$.

• First, let us show the first part of the theorem.

— For $k = 0$, we have $I_k = I_0 = [1, a)$. According to the definition of ℓ, we have

$$\Phi^{\ell} \leq a < \Phi^{\ell+1}.$$

Hence

$$\bigsqcup_{i=0}^{\ell-1} \left[\Phi^i, \Phi^{i+1}\right) \subset I_0 \subset \bigsqcup_{i=0}^{\ell} \left[\Phi^i, \Phi^{i+1}\right) \tag{2.5}$$

(recall that the symbol $\sqcup$ denotes a disjoint union). According to Lemma 2.1, since each interval $[\Phi^i, \Phi^{i+1})$ $(i \in \mathbb{N})$ contains a unique Fibonacci number, it follows from (2.5) that the interval I_0 contains at least ℓ Fibonacci numbers and at most $(\ell + 1)$ Fibonacci numbers, as required.

— For the following, we assume $k \geq 1$. Let i denote the number of the Fibonacci numbers belonging to I_k and let $F_r, F_{r+1}, \ldots, F_{r+i-1}$ $(r \geq 2)$ denote those Fibonacci numbers. We shall determine i. By definition, we have

$$F_{r-1} < a^k \leq F_r < F_{r+1} < \cdots < F_{r+i-1} < a^{k+1} \leq F_{r+i} \tag{2.6}$$

which implies that

$$\frac{F_{r+i-1}}{F_r} < \frac{a^{k+1}}{a^k} = a \tag{2.7}$$

and

$$\frac{F_{r+i}}{F_{r-1}} > \frac{a^{k+1}}{a^k} = a. \tag{2.8}$$

On the other hand, we have

$$\begin{aligned} F_{r+i-1} &< a^{k+1} && \text{(according to (2.6))} \\ &\leq a^{2k} && \text{(since } k \geq 1\text{)} \\ &\leq F_r^2 && \text{(according to (2.6))} \\ &\leq F_{2r-1} && \text{(according to Lemma 2.4).} \end{aligned}$$

Hence $F_{r+i-1} < F_{2r-1}$. Because the sequence $(F_n)_{n\in\mathbb{N}}$ is non-decreasing, we deduce that $r+i-1 < 2r-1$, which gives $r > i$, that is, $r \geq i+1$. This then allows us to apply Lemma 2.6 for each of the two couples $(n,m) = (r, i-1)$ and $(n,m) = (r-1, i+1)$ to obtain

$$\left\lfloor \Phi^{i-1} \right\rfloor \;\leq\; \frac{F_{r+i-1}}{F_r} \;\leq\; \left\lceil \Phi^{i-1} \right\rceil$$

and

$$\left\lfloor \Phi^{i+1} \right\rfloor \;\leq\; \frac{F_{r+i}}{F_{r-1}} \;\leq\; \left\lceil \Phi^{i+1} \right\rceil.$$

By comparing these last double inequalities with (2.7) and (2.8), we deduce that

$$\left\lfloor \Phi^{i-1} \right\rfloor < a < \left\lceil \Phi^{i+1} \right\rceil.$$

But since a is an integer, it follows that

$$\Phi^{i-1} < a < \Phi^{i+1},$$

which gives

$$\frac{\log a}{\log \Phi} - 1 < i < \frac{\log a}{\log \Phi} + 1.$$

Finally, since i is an integer, we conclude that

$$i \in \left\{ \left\lfloor \frac{\log a}{\log \Phi} \right\rfloor,\; \left\lceil \frac{\log a}{\log \Phi} \right\rceil \right\} = \{\ell, \ell+1\},$$

as required.

• Now, let us show the second part of the theorem which deals with densities of subsets of the natural numbers. For a given positive integer N, the intervals $I_k := [a^k, a^{k+1})$ $(0 \leq k \leq N-1)$ clearly form a partition of the interval $[1, a^N)$. Let A_N denote the number of the intervals I_k $(0 \leq k \leq N-1)$, each of them contains exactly ℓ Fibonacci numbers and let B_N denote the number of the intervals I_k $(0 \leq k \leq N-1)$, each of them contains exactly $(\ell+1)$ Fibonacci numbers. According to the first part of the theorem (shown above), we have

$$A_N + B_N = N$$

and

$$\ell A_N + (\ell+1) B_N = \operatorname{Card}\left(\mathscr{F} \cap [1, a^N)\right) \sim_{+\infty} \frac{\log(a^N)}{\log \Phi} = \frac{\log a}{\log \Phi} N$$

(where the last estimate follows from Lemma 2.3).

So, the couple (A_N, B_N) is a solution of the following linear system of two equations

$$\begin{cases} A_N + B_N = N, \\ \ell A_N + (\ell+1) B_N = \dfrac{\log a}{\log \Phi} N + o(N). \end{cases}$$

By solving this system, we get

$$\begin{aligned} A_N = \left(\ell + 1 - \frac{\log a}{\log \Phi}\right) N + o(N) &= \left(\left\lfloor \frac{\log a}{\log \Phi} \right\rfloor + 1 - \frac{\log a}{\log \Phi}\right) N + o(N) \\ &= \left(1 - \left\langle \frac{\log a}{\log \Phi} \right\rangle\right) N + o(N) \end{aligned}$$

and

$$\begin{aligned} B_N = \left(\frac{\log a}{\log \Phi} - \ell\right) N + o(N) &= \left(\frac{\log a}{\log \Phi} - \left\lfloor \frac{\log a}{\log \Phi} \right\rfloor\right) N + o(N) \\ &= \left\langle \frac{\log a}{\log \Phi} \right\rangle N + o(N). \end{aligned}$$

Hence

$$\lim_{N \to +\infty} \frac{A_N}{N} = 1 - \left\langle \frac{\log a}{\log \Phi} \right\rangle \qquad \text{and} \qquad \lim_{N \to +\infty} \frac{B_N}{N} = \left\langle \frac{\log a}{\log \Phi} \right\rangle.$$

These two limits respectively represent the density of the set of all $k \in \mathbb{N}$ for which the interval I_k contains exactly $\ell = \left\lfloor \frac{\log a}{\log \Phi} \right\rfloor$ Fibonacci numbers and the density of the set of all $k \in \mathbb{N}$ for which I_k contains exactly $(\ell + 1) = \left\lceil \frac{\log a}{\log \Phi} \right\rceil$ Fibonacci numbers. This confirms the second part of the theorem and completes this proof. □

3. Numerical examples and remarks

In this section, we apply our main result for some particular values of a to deduce some interesting results.

- For $a = 10$, the first part of Theorem 1.1 shows that any interval of the form $[10^k, 10^{k+1})$ $(k \in \mathbb{N})$ contains either $\left\lfloor \frac{\log 10}{\log \Phi} \right\rfloor = 4$ or $\left\lceil \frac{\log 10}{\log \Phi} \right\rceil = 5$ Fibonacci numbers. We thus find again Lamé's result cited in Section 1. The second part of Theorem 1.1 shows that the set of all positive integers k for which there are exactly 4 Fibonacci numbers with k digits and the set of all positive integers k for which there are exactly 5 Fibonacci numbers with k digits have the density $1 - \left\langle \frac{\log 10}{\log \Phi} \right\rangle = 0.215\ldots$ and $\left\langle \frac{\log 10}{\log \Phi} \right\rangle = 0.784\ldots$, respectively.
- For $a = 7$, Theorem 1.1 shows that any interval of the form $[7^k, 7^{k+1})$ $(k \in \mathbb{N})$ contains either 4 or 5 Fibonacci numbers. Besides, the density of the set of all $k \in \mathbb{N}$ for which the interval $[7^k, 7^{k+1})$ contains exactly 4 Fibonacci numbers is about 0.956. So, there is more than a 95% chance that an arbitrary interval of the form $[7^k, 7^{k+1})$ contains exactly 4 Fibonacci numbers.

Remark 1. The second example above shows that for $a = 7$, the intervals $[a^k, a^{k+1})$ $(k \in \mathbb{N})$ contain almost all (i.e., with a large percentage) the same quantity of Fibonacci numbers. Interestingly, we remark that $7 = L_4$ is a Lucas number. Actually, it is not difficult to show that the last property is satisfied for any other Lucas number $a \geq 7$. Indeed, if a is a Lucas number (say $a = L_n$ for some $n \geq 4$), then a is close to Φ^n (since $L_n = \Phi^n + \overline{\Phi}^n$, according to (1.8)), and then $\frac{\log a}{\log \Phi}$ is close to n. It follows that one of the two densities occurring in Theorem 1.1 (that is $\langle \frac{\log a}{\log \Phi} \rangle$ and $1 - \langle \frac{\log a}{\log \Phi} \rangle$) is almost zero, which confirms that the intervals $[a^k, a^{k+1})$ $(k \in \mathbb{N})$ contain almost all same quantity of Fibonacci numbers. Taking, for example, $a = L_{11} = 199$, we find that more than 99.99% of the intervals $[199^k, 199^{k+1})$ $(k \in \mathbb{N})$ contain exactly 11 Fibonacci numbers. The few remaining of these intervals contain only 10 Fibonacci numbers.

Remark 2. Let $a \geq 2$ be a fixed integer. For an arbitrary natural number k, let X denote the random variable that counts the number of the Fibonacci numbers belonging to the interval $[a^k, a^{k+1})$. Theorem 1.1 shows that X takes only two possible values $\lfloor \frac{\log a}{\log \Phi} \rfloor$ and $\lceil \frac{\log a}{\log \Phi} \rceil$ with the probabilities $(1 - \langle \frac{\log a}{\log \Phi} \rangle)$ and $\langle \frac{\log a}{\log \Phi} \rangle$, respectively. Using this, the calculations give

$$\mathbb{E}(X) = \frac{\log a}{\log \Phi}$$

$$\sigma(X) = \sqrt{\Big\langle \frac{\log a}{\log \Phi} \Big\rangle \Big(1 - \Big\langle \frac{\log a}{\log \Phi} \Big\rangle\Big)},$$

where $\mathbb{E}(X)$ and $\sigma(X)$, respectively, denote the mathematical expectation and the standard deviation of X.

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